On Tempered and Substantial Fractional Calculus

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***Abstract*—In this paper, we discuss the differences between the tempered fractional calculus and substantial fractional operators in anomalous diffusion modelling, so that people can better understand the two fractional operators. We first introduce the definitions of tempered and substantial fractional operators, and then analyze the properties of two definitions. At last, we prove that the tempered fractional derivative and substantial derivative are equivalent under some conditions. A diffusion problem defined by using tempered derivative is also given to illustrate the slow convergence of an anomalous diffusion process.**

***Keywords*—Fractional calculus, Tempered fractional calculus, Substantial fractional calculus, Anomalous diffusion**

I. Introduction

The process of anomalous diffusion is one of common phenomena in nature, and the continuous time random walks (CTRWs) framework [1] is a useful tool to describe this phenomenon. The CTRWs are often governed by the waiting time probability density function (PDF) and jump length PDF. When the PDFs are power law, the anomalous transport process can be depicted by fractional diffusion equations. While the PDFs are exponentially tempered power law, then tempered anomalous diffusion models are derived in [2–8]. As Meerschaert [9] pointed out, tempered stable processes are the limits of random walk models where the power law probability of long jumps is tempered by an exponential factor. These random walks converge to tempered stable stochastic process limits, whose probability densities solve tempered fractional diffusion equations. Tempered power law waiting times lead to tempered fractional time derivatives, which have proven useful in geophysics. Meerschaert et al. proposed a tempered diffusion model to capture the slow convergence of subdiffusion [6].

Baeumer and Meerschaert studied tempered stable Lévy motion in [2], they proposed finite difference and particle tracking methods to solve the tempered fractional diffusion equation with drift. In view of the efficiency of tempered fractional calculus in describing exponentially tempered power law behavior and its variants, it has attracted many researchers to study numerical methods to solve these problems. Baeumera and Meerschaert [2] derived finite difference and particle tracking methods. Cartea *et al.* [10] presented a general finite difference scheme to numerically solve a Black-Merton-Scholes model with tempered fractional derivatives. Momoniat and Momoniat [11] compare the numerical solutions of three kinds of fractional Black-Merton-Scholes equations with tempered fractional derivatives. Recently, high order numerical scheme for tempered diffusion equation is presented in [12]. However, numerical algorithms for solving these problems are limited.

As an extension of the concept of CTRWs to phase space, Friedrich *et al.* derived a new fractional Kramers–Fokker–Planck equation [13], which involved a fractional substantial derivative, it has important nonlocal couplings in both time and space. In 2011, based on the CTRW models with coupling PDFs, Carmi and Barkai obtained a deterministic equation by using fractional substantial derivative [14]. The properties and numerical discretizations of the fractional substantial operators are recently discussed in [15]. To our best knowledge , whether the tempered fractional operators [3] or the fractional substantial operators [13] is originated from the tempered function space. Motivated by this, we try to let people know the relationship of these two fractional operators.

The work is organized as follows. In Section II, we introduce three common used definitions of fractional integrals and derivatives. Two classes of fractional operators called tempered and substantial operators are introduced in Section III. Numerical experiment is carried out to show the effectiveness of tempered model in describing exponentially tempered power-law behavior. Finally, we conclude the paper in the last section.

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II. Preliminaries

In this section, we give some preliminaries about fractional calculus. There are sveral different definitions of fractional derivatives, but the most frequently used are the following three definitions, i.e. Grünwald–Letnikov derivative, the Riemann–Liouville derivative and the Caputo derivative [16–21]. We introduce the definitions in the following way.

Definition II.1. *The fractional integral of order $\alpha > 0$ for a function $f(t)$ is defined by*

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \tag{1}$$

where $\Gamma(\cdot)$ is the Euler's function.

Definition II.2. *The left and right Grünwald–Letnikov derivatives of order $\alpha > 0$ of $f(t)$ are defined as*

$$ {}_{GL}D_{a,t}^{\alpha} f(t) = \lim_{\substack{h\to 0\\ Nh=t-a}} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{2}$$

and

$$ {}_{GL}D_{t,b}^{\alpha} f(t) = \lim_{\substack{h\to 0\\ Nh=b-t}} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t+jh), \quad (3) $$

respectively.

Definition II.3. *Suppose that $f(t)$ be $(n-1)$-times continuously differentiable on (a,∞), and its n-times derivatives be integrable on any subinterval $[a,\infty)$. Then the left Riemann-Liouville derivative of order $\alpha>0$ of $f(t)$ is defined by*

$$ \begin{aligned} {}_{RL}D_{a,t}^{\alpha} f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[{}_aD_t^{-(n-\alpha)} f(t)\right] \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_a^t (t-s)^{n-\alpha-1} f(s)\,\mathrm{d}s, \end{aligned} \quad (4) $$

and the right Riemann-Liouville fractional derivative is defined as

$$ {}_{RL}D_{t,b}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^b (s-t)^{n-\alpha-1} f(s)\,\mathrm{d}s, \quad (5) $$

respectively, where n is a nonnegative integer and $n-1\le\alpha<n$.

Definition II.4. *Assume that $f(t)$ be $(n-1)$-times continuously differentiable on (a,∞), and its n-times derivatives be integrable on any subinterval $[a,\infty)$. Then the left Caputo fractional derivative of order $\alpha>0$ for $f(t)$ is defined as*

$$ \begin{aligned} {}_{C}D_{a,t}^{\alpha} f(t) &= {}_aD_t^{-(n-\alpha)}\left[f^{(n)}(t)\right] \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \end{aligned} \quad (6) $$

and the right Caputo derivative is defined by

$$ {}_{C}D_{t,b}^{\alpha} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s, \quad (7) $$

respectively, where n is a nonnegative integer and $n-1<\alpha<n$.

III. DEFINITIONS OF TEMPERED AND SUBSTANTIAL FRACTIONAL CALCULUS

In this section, we will introduce the definitions and notations of tempered and substantial fractional operators. Then we discuss the relations between them.

A. Definitions of the tempered fractional operators

Definition III.1. *[3, 12] Suppose that $f(t)$ is piecewise continuous on $[a,\infty)$ and integrable on any finite subinterval of $[a,\infty)$, $\alpha>0$, $\lambda\ge 0$. Then*
1) The left Riemann-Liouville tempered fractional integral of order α of function $f(t)$ is defined by

$$ {}_{RL}D_{a,t}^{-\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t e^{-\lambda(t-\tau)}(t-\tau)^{\alpha-1} f(\tau)d\tau. $$

2) The right Riemann-Liouville tempered fractional integral of order α for $f(t)$ is defined as

$$ {}_{RL}D_{b,t}^{-\alpha,\lambda} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b e^{-\lambda(\tau-t)}(\tau-t)^{\alpha-1} f(\tau)d\tau. $$

Definition III.2. *[3, 12] Let $f(t)$ be $(n-1)$-times continuously differentiable on (a,∞), and its n-times derivatives be integrable on any subinterval $[a,\infty)$. Then the left tempered fractional derivative of order $\alpha>0$ for a given function $f(t)$ is defined as*

$$ \begin{aligned} {}_{RL}D_{a,t}^{\alpha,\lambda} f(t) &= (e^{-\lambda t}\,{}_{RL}D_{a,t}^{\alpha} e^{\lambda t})f(t) \\ &= \frac{e^{-\lambda t}}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1} e^{\lambda\tau} f(\tau)d\tau, \end{aligned} \quad (8) $$

and the right tempered fractional derivative is defined as

$$ \begin{aligned} {}_{RL}D_{t,b}^{\alpha,\lambda} f(t) &= (e^{\lambda t}\,{}_{RL}D_{t,b}^{\alpha} e^{-\lambda t})f(t) \\ &= \frac{(-1)^n e^{\lambda t}}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (\tau-t)^{n-\alpha-1} e^{-\lambda\tau} f(\tau)d\tau, \end{aligned} \quad (9) $$

respectively, where n is a nonnegative integer and $n-1\le\alpha<n$.

Remark III.1. *If $\lambda=0$, the left and right Riemann-Liouville tempered fractional derivatives reduce to the left and right Riemann-Liouville fractional derivatives defined in Definition II.3.*

Remark III.2. *The variants of the left and right Riemann-Liouville tempered fractional derivatives are defined as [2, 12, 22]*

$$ {}_{RL}\mathbf{D}_{a,t}^{\alpha,\lambda} f(t) = \begin{cases} {}_{RL}D_{a,t}^{\alpha,\lambda} f(t) - \lambda^{\alpha} f(t), \ 0<\alpha<1, \\ {}_{RL}D_{a,t}^{\alpha,\lambda} f(t) - \alpha\lambda^{\alpha-1}\partial_t f(t) - \lambda^{\alpha} f(t), 1<\alpha<2, \end{cases} \quad (10) $$

and

$$ {}_{RL}\mathbf{D}_{t,b}^{\alpha,\lambda} f(t) = \begin{cases} {}_{RL}D_{t,b}^{\alpha,\lambda} f(t) - \lambda^{\alpha} f(t), \ 0<\alpha<1, \\ {}_{RL}D_{t,b}^{\alpha,\lambda} f(t) - \alpha\lambda^{\alpha-1}\partial_t f(t) - \lambda^{\alpha} f(t), 1<\alpha<2, \end{cases} \quad (11) $$

respectively, where ∂_t represents the classical first derivative operator.

B. Substantial fractional operators

Definition III.3. *[13, 15] Let $f(t)$ be piecewise continuous on $[a,\infty)$ and integrable on any finite subinterval of $[a,\infty)$. Then the fractional substantial integral of order $\alpha>0$ for $f(t)$ is defined by*

$$ D_s^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t e^{-\lambda(t-\tau)}(t-\tau)^{\alpha-1} f(\tau)d\tau, $$

where λ can be a constant or a function not related to t.

Definition III.4. *[13, 15] Suppose that $\alpha>0$, $f(t)$ be $(n-1)$-times continuously differentiable on (a,∞), and its n-times derivatives be integrable on any subinterval $[a,\infty)$. Then the* *substantial fractional derivative of order $\alpha > 0$ for $f(t)$ is defined by*

$$D_s^\alpha f(t) = D_s^n \left(D_s^{-(n-\alpha)} f(t) \right),$$

where $D_s^n = \left(\frac{d}{dt} + \lambda\right)^n$.

Remark III.3. *If $\lambda \geq 0$, it is clear that Definition III.1 is equivalent to Definition III.3.*

Theorem III.1. *In Definition III.4, if λ is a positive constant, then the tempered and substantial derivatives are equivalent.*

Proof: Without loss of generality, we take $0 < \alpha < 1$ in Definition (8) and Definition III.4, then the left Riemann-Liouville tempered derivative is

$$_{RL}D_{a,t}^{\alpha,\lambda} f(t) = \frac{e^{-\lambda t}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t e^{\lambda\tau} (t-\tau)^{-\alpha} f(\tau) d\tau, \quad (12)$$

and the substantial derivative becomes

$$\begin{aligned} D_s^\alpha f(t) &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dt} + \lambda\right) \int_a^t e^{-\lambda(t-\tau)} (t-\tau)^{-\alpha} f(\tau) d\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(e^{-\lambda t} \int_a^t e^{\lambda\tau} (t-\tau)^{-\alpha} f(\tau) d\tau \right) \\ &\quad + \frac{\lambda}{\Gamma(1-\alpha)} \int_a^t e^{-\lambda(t-\tau)} (t-\tau)^{-\alpha} f(\tau) d\tau \\ &= \frac{-\lambda}{\Gamma(1-\alpha)} \int_a^t e^{-\lambda(t-\tau)} (t-\tau)^{-\alpha} f(\tau) d\tau \\ &\quad + \frac{e^{-\lambda t}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t e^{\lambda\tau} (t-\tau)^{-\alpha} f(\tau) d\tau \\ &\quad + \frac{\lambda}{\Gamma(1-\alpha)} \int_a^t e^{-\lambda(t-\tau)} (t-\tau)^{-\alpha} f(\tau) d\tau \\ &= \frac{e^{-\lambda t}}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t e^{\lambda\tau} (t-\tau)^{-\alpha} f(\tau) d\tau. \end{aligned} \quad (13)$$

The proof ends. ■

Definition III.5. *[23, 24] The left and right generalized fractional integral of order $\alpha > 0$ of a function $f(t)$ with respect to another function $z(t)$ and weight $w(t)$ are defined in the following way*

$$\left(I_{a,+;[z;w]}^\alpha f\right)(t) = \frac{[w(t)]^{-1}}{\Gamma(\alpha)} \int_a^t \frac{w(\tau) z'(\tau) f(\tau)}{[z(t) - z(\tau)]^{1-\alpha}} d\tau, \quad (14)$$

and

$$\left(I_{b,-;[z;w]}^\alpha f\right)(t) = \frac{[w(t)]}{\Gamma(\alpha)} \int_t^b \frac{w(\tau) z'(\tau) f(\tau)}{[z(\tau) - z(t)]^{1-\alpha}} d\tau, \quad (15)$$

respectively.

Remark III.4. *If we take $z(t) = t$, $w(t) = e^{\lambda t}$, then the left and right generalized integrals reduce to the left and right tempered fractional integrals.*

IV. Numerical simulation

In this section, based on the discussion of tempered and substantial derivatives, we use finite difference method to solve a tempered diffusion problem.

Example IV.1. *Solve the following tempered fractional diffusion equation*

$$\frac{\partial u(x,t)}{\partial t} = {}_{RL}D_{0,x}^{0.5,\lambda} u(x,t) + f(x,t),\ 0 < x < 1,\ 0 < t < 1, \quad (16)$$

with the initial and boundary conditions: $u(x,0) = 0$, $u(0,t) = 0$, $u(1,t) = e^{-\lambda x}$, where $f(x,t) = 2e^{-\lambda x} x^{2.5} t - \frac{\Gamma(3.5)}{2} x^2 e^{-\lambda x} t^2$. The analytical solution of Eq. (16) is

$$u(x,t) = e^{-\lambda x} x^{2.5} t^2.$$

For the numerical solution of Eq. (16), let $t_k = k\Delta t$, $k = 0,1,2,...,N$, $x_i = ih$, $i = 0,1,2,...,M$, where $\Delta t = \frac{T}{N}$ and $h = \frac{L}{M}$ are the time, space steps, respectively.

The first-order derivative $\frac{\partial u(x,t)}{\partial t}$ at mesh point (x_i, t_k) can be approximated by the following backward difference method

$$\frac{\partial u(x_i,t_k)}{\partial t} = \frac{u(x_i,t_k) - u(x_i,t_{k-1})}{\Delta t} + O(\Delta t), \quad (17)$$

then, we use shifted Grünwald–Letnikov formulae to approximate fractional substantial derivative term in Eq. (16)

$$D_s^\alpha u(x_i,t_k) = h^{-\alpha} \sum_{m=0}^{i+1} g_m^{1,\alpha} u(x_{i-m+1},t_k) + O(h). \quad (18)$$

Let u_i^k be the approximate solution of $u(x_i,t_k)$, and $f_i^k = f(x_i,t_k)$, substituting the Eqs. (17) and (18) into Eq. (16), and denoting $\eta = K\frac{\Delta t}{h^\alpha}$, we obtain the following implicit finite difference scheme for Eq. (16)

$$\begin{cases} u_i^k = u_i^{k-1} + \eta \sum_{m=0}^{i+1} g_m^{1,\alpha} u_{i-m+1}^k \\ \qquad + \Delta t f_i^k,\ 1 \leq i \leq M-1,\ 1 \leq k \leq N-1 \\ u_i^0 = 0,\ 0 \leq i \leq M, \\ u_0^k = \phi_1(t_k),\ u_M^k = \phi_2(t_k),\ 0 \leq k \leq N. \end{cases} \quad (19)$$

Theorem IV.1. *The local truncation error of difference scheme (19) is $O(\tau + h)$.*

Proof: According to (16), (17) and (18), we define the local truncation error R_i^k of difference scheme (19) as below:

$$\begin{aligned} R_i^k &= \frac{u(x_i,t_k) - u(x_i,t_{k-1})}{\Delta t} - Kh^{-\alpha} \sum_{m=0}^{i+1} g_m^{1,\alpha} u(x_{i-m+1},t_k) \\ &\quad - f(x_i,t_k) \\ &= \left[\frac{\partial u(x_i,t_k)}{\partial t} - \frac{u(x_i,t_k) - u(x_i,t_{k-1})}{\Delta t} \right] \\ &\quad + K(D_s^\alpha u(x_i,t_k) - h^{-\alpha} \sum_{m=0}^{i+1} g_m^{1,\alpha} u(x_{i-m+1},t_k)) \\ &= O(\tau) + KO(h) = O(\tau + h). \end{aligned}$$

The proof ends.

■

Let $\lambda = 0,\ 0.5,\ 1.0$, the analytical and numerical solutions are displayed in Fig. 1. It can be seen that the numerical solutions fit the analytical solutions very well. When $\lambda = 0$, the equation (16) reduces to the Riemann–Liouville diffusion equation, Fig. 1 (a) and (b) show that solution peak is high. For $\lambda = 0.5$ and $\lambda = 1.0$, the solution are plotted in Fig. 1(c), (d) and Fig. 1(e), (f), respectively. From Fig. 1, we can see that the peak of the solutions of tempered diffusion equation becomes more and more smooth as exponential factor λ increases.

V. Conclusion

In this paper, we introduce two classes of fractional operators for anomalous diffusion, and further discuss the properties of tempered and substantial derivatives. We obtain a theorem on two definitions under some conditions. It is easy to conclude that tempered and substantial fractional calculus are the generalization of fractional calculus, and both of them are special cases of generalized fractional calculus. Although substantial derivative is equivalent to tempered derivative when the parameter $\lambda \geq 0$, they are introduced from different physical backgrounds. Mathematically the fractional substantial calculus is time-space coupled operator but the tempered fractional calculus is not. However, the tempered fractional operators are the more commonly used in truncated exponential power law description.

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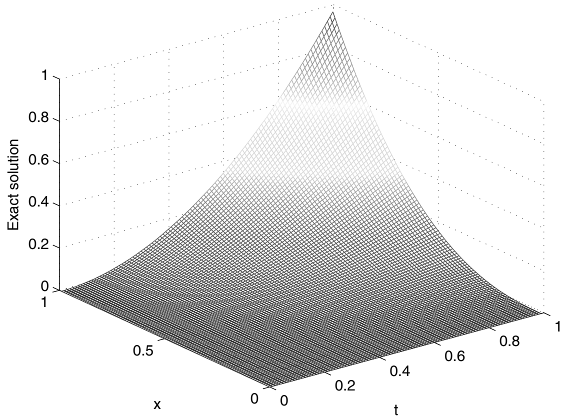
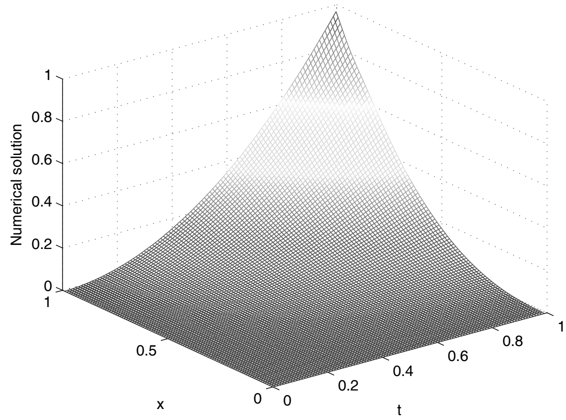


(a)



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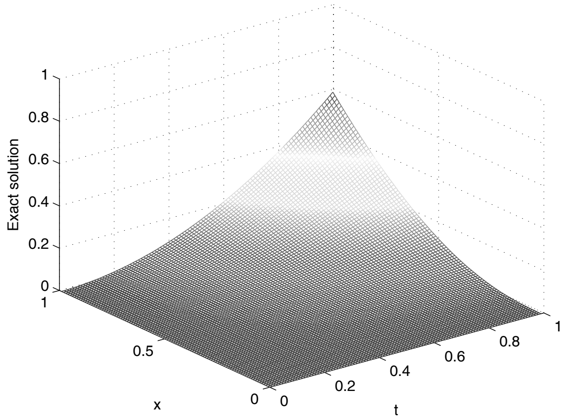

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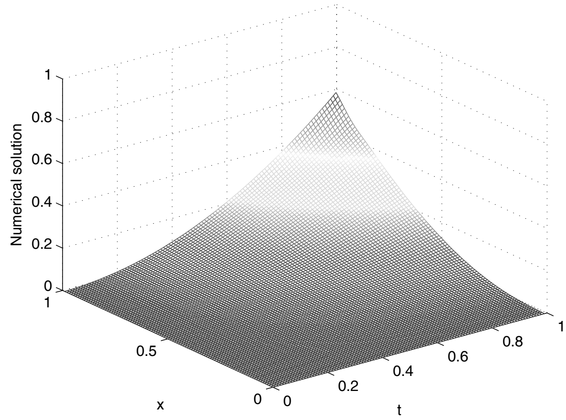

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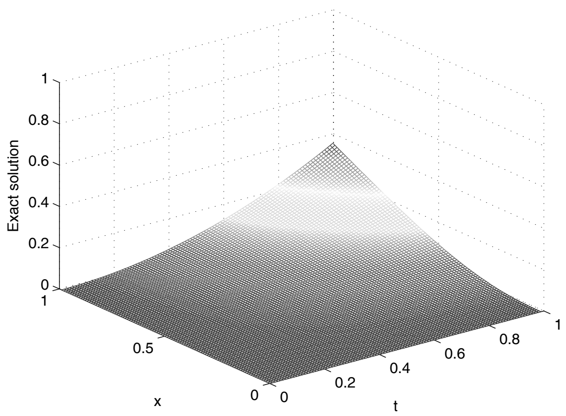

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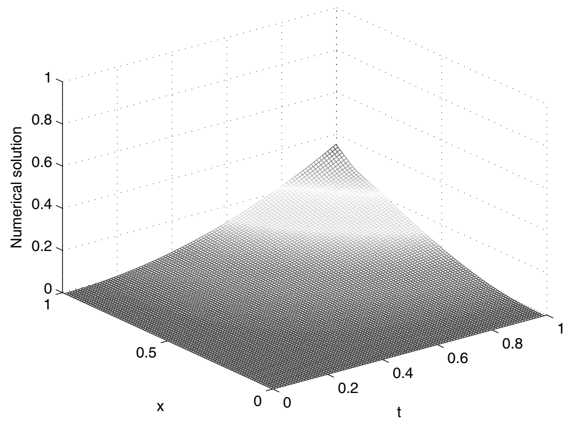

(f)

Fig. 1. The comparision between exact and numerical solutions

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