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Brief paper

Fractional-order sliding mode based extremum seeking control of a class of nonlinear systems☆

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ABSTRACT

This paper develops a fractional-order sliding mode based extremum seeking controller (FO SM-ESC) for the optimization of nonlinear systems. The proposed FO SM-ESC, involving an FO derivative function ${}_0D_t^q \mathrm{sgn}(e)$, $0 \le q < 1$ is shown to achieve faster tracking and higher control accuracy than the integer-order (IO) SM-ESC. The tradeoff between control performance and parameters selection is analyzed and visualized. The comparison between the FO/IO SM-ESC is given to reveal the potential advantages of the FO controller over the IO controller. Simulation and experimental results show that the FO SM-ESC can have a faster convergence speed and a smaller neighborhood around the optimal operational point.

1. Introduction

1.1. Background

The desired operational point of a considered process should happen in the optimum of a nonlinear performance that is always connected to efficiency of plant in a lot of applications. The ESC (Ariyur & Krstić, 2003; DeHaan & Guay, 2005; Krstić & Wang, 2000; Popovic, Jankovic, Magner, & Teel, 2006; Tan, Nešić, & Mareels, 2006; Teel & Popović, 2001; Zhang & Ordóñez, 2009) is a method for tracking a variational optimum. ESC and its applications have been extensively studied, such as maximum power point tracking (MPPT) for the photovoltaic (PV) generators (Malek, Dadras, & Chen, 2012), and source seeking (Cochran & Krstić, 2007). One popular technique in ESC is sliding mode based method (Drakunov, Özgüner, Dix, & Ashrafi, 1995; Fu & Özgüner, 2011; Oliveira, Peixoto, & Hsu, 2012; Pan, Özgüner, & Acarman, 2003; Yin, Stark, Zhong, & Chen, 2012). Drakunov et al. (1995) have developed an SM-ESC for the output optimization of dynamic systems.

Extending IO derivatives and integrals to FO items has a solid standing foundation. Applying FO controllers to IO systems, there is a better flexibility in regulating FO than utilizing IO controllers. The possible benefits have activated renewed interest in varieties of FO control (Luo & Chen, 2012; Malek et al., 2012; Podlubny, 1999; Yin, Stark, Chen, & Zhong, 2013). For example, Luo et al. in Luo and Chen (2012) have shown the advantage of the FOPI over the IOPID.

1.2. Summary of contributions

In this paper, some useful tools for FO control analysis of IO systems are given. A concept of the FO sign function ${}_0D_t^q \mathrm{sgn}(e)$, $0 \le q < 1$, involving an FO differentiator, is introduced. ${}_0D_t^q \mathrm{sgn}(e)$ is proven to be able to extract the sign of e. One may feel this is trivial compared with the sign function itself; others may doubt that this is against intuition compared with the derivative of a generic function. *The meaning of* ${}_0D_t^q \mathrm{sgn}(e)$, $0 \le q < 1$ *is the fractional order derivative of the sign function. The sign of the derivative is generally not the same as the sign of the function itself. However, the sign of* ${}_0D_t^q \mathrm{sgn}(e)$, $0 \le q < 1$ *is firstly proven to be the same as* $\mathrm{sgn}(e)$. This is the first important property of the FO sign function. In addition,

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we show that the absolute value of ${}_0D_t^q\mathrm{sgn}(e)$ can be greater than 1 (i.e. $|\mathrm{sgn}(e)|$ is just limited at 0 or 1). This is the second important property of the FO sign function.

Based on ${}_0D_t^q\mathrm{sgn}(e)$, an FO SM-ESC is proposed for the optimization of nonlinear systems. In the FO SM-ESC, the first property of ${}_0D_t^q\mathrm{sgn}(e)$, $0 \le q < 1$ guarantees that the output achieves an arbitrary small neighborhood of the optimal operational point and remains close to it thereafter. The second property can improve the control performance (i.e. convergence speed and accuracy). The changes of ${}_0D_t^q\mathrm{sgn}(e)$, $0 \le q < 1$ with respect to q and t are visualized to show how the properties of the FO sign function contribute to the superior tracking behavior. Further FO/IO SM-ESC comparisons are given. They show the advantage of the FO SM-ESC over the IO SM-ESC clearly. Both simulation and experimental results demonstrate the advantages of the designed control scheme.

2. Fractional-order sliding mode based extremum seeking control

2.1. Problem formulation

Consider a single-input–single-output (SISO) nonlinear system (Krstić & Wang, 2000; Pan et al., 2003)

$$\frac{d}{dt}x = f(x,u), \tag{1}$$

$$y = \tilde{F}(x), \tag{2}$$

where $x \in R^n$, $u \in R, y \in R$ are respectively the state, control input and output. $f(x,u)$ and $\tilde{F}(x)$ are smooth. Without loss of generality, we consider the maximization of y. We do not assume the knowledge of $\tilde{F}(\cdot)$. Consider a smooth control law: $u = \eta(x,\theta)$, parameterized by a scalar parameter θ. The system (1) is derived as $\frac{d}{dt}x = f(x,\eta(x,\theta))$.

Assumption 1. $\exists$ a smooth function $x_e : R \to R^n$ such that $f(x,\eta(x,\theta)) = 0$ if and only if $x = x_e(\theta)$.

Assumption 2. At the equilibrium point $x_e(\theta)$, the static performance map from θ to y (i.e. $y(t) = \tilde{F}(x_e(\theta)) = F(\theta)$) is smooth.

We suppose that there exists a unique maximum at θ^* for $F(\theta)$. θ^*, $F(\cdot)$ and its gradient for any θ are assumed to be unknown to the control designer. By utilizing $F'(\theta) = dF(\theta)/d\theta$, $F''(\theta) = d^2F(\theta)/d\theta^2$, one has $\frac{d}{dt}y = F'(\theta)\frac{d}{dt}\theta$. The following assumption is listed.

Assumption 3. There exists a unique $\theta^* \in R$ such that $F'(\theta^*) = 0$ and $F''(\theta^*) < 0$. For any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $|F'(\theta)| > \varepsilon$, $\forall \theta \notin D_\delta$, where $D_\delta = \{\theta : |\theta - \theta^*| < \delta/2\}$ is called δ-vicinity of θ^*.

In order to design an FO SM-ESC, we use the following definition and lemmas.

Definition 1. The Riemann–Liouville (RL) definition of the qth order derivative is Monje, Chen, Vinagre, Xue, and Feliu (2010) and Samko, Kilbas, and Marichev (1993) ${}_0D_t^q f(t) = \frac{d^q f(t)}{dt^q} = \frac{1}{\Gamma(m-q)}\frac{d^n}{dt^n}\int_0^t \frac{f(\iota)}{(t-\iota)^{q+1-n}}d\iota$, where $n-1 < q \le n$ with n is the integer, $\Gamma(\cdot)$ is the gamma function.

Lemma 1 (*Kilbas, Srivastava, & Trujillo, 2006 and Samko et al., 1993*)**.** *The fractional integration operator* ${}_aI_t^\beta = \frac{1}{\Gamma(\beta)}\int_a^t \frac{f(\iota)}{(t-\iota)^{1-\beta}}d\iota$, $(t > a; \beta \in C, Re(\beta) > 0)$ *is bounded in* $L_p(\hat{a},\hat{b})$, $(1 \le p \le \infty, -\infty < \hat{a} < \hat{b} < \infty)$*:*

$$\|{}_aI_t^\beta f\|_p \le K\|f\|_p, \qquad \left(K = \frac{(\hat{b}-\hat{a})^{\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)|\Gamma(\beta)|}\right). \tag{3}$$

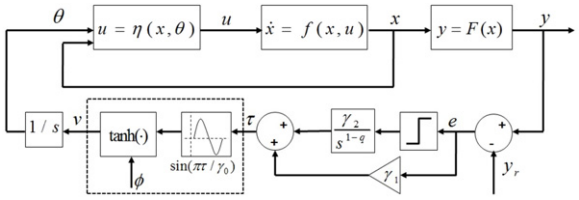

Fig. 1. Block diagram of the proposed FO SM-ESC scheme.

Lemma 2. *Consider the RL fractional derivative operator* ${}_0D_t^q f(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^q}d\tau$, $0 \le q < 1$ *and sign function, one can obtain*

$${}_0D_t^q\mathrm{sgn}(e(t)) \begin{cases} > 0, & \text{if } e(t) > 0, t > 0, \\ < 0, & \text{if } e(t) < 0, t > 0. \end{cases} \tag{4}$$

Proof. See Appendix A.

From Lemma 2, ${}_0D_t^q\mathrm{sgn}(e)$, $0 \le q < 1$ is proven to be able to extract the sign of e. This is its first property.

Remark 1. Consider two cases: (1) $e(t) > 0$, $\forall t > 0$, one has ${}_0D_t^q\mathrm{sgn}(e) = \frac{t^{-q}}{\Gamma(1-q)}$; (2) $e(t) < 0$, $\forall t > 0$, one also has ${}_0D_t^q\mathrm{sgn}(e) = \frac{-t^{-q}}{\Gamma(1-q)}$. Thus, ${}_0D_t^q\mathrm{sgn}(e)$ are bounded in both cases. Since the absolute value of the above cases is greater than the other cases, ${}_0D_t^q\mathrm{sgn}(e)$, $\forall e$ is bounded, i.e. there exists $\varpi > 0$ such that $|{}_0D_t^q\mathrm{sgn}(e)| < \varpi$.

2.2. FO sliding mode based extremum seeking control

An FO SM-ESC will be designed for the system (1). First, the tracking error is defined as $e = y - y_r$, in which $\dot{y}_r = k_r$, $y_r(0) = y_{r0}$, with $k_r > 0$ and y_{r0} is a design constant. So as to avoid an unbounded $y_r(t)$, one can saturate $y_r(t)$ at a rough known upper bound of y^*. Based on e, a function τ is proposed by

$$\tau = \gamma_1 e + \gamma_2({}_0D_t^{q-1}\mathrm{sgn}(e)), \tag{5}$$

where $\gamma_i > 0$, $(i = 1, 2)$, $0 \le q < 1$, ${}_0D_t^{q-1}$ is an RL fractional operator. The parameter θ is defined to meet $\dot{\theta} = v$, in which v denotes variable structure control

$$v = \phi\tanh\left(\sin\left(\frac{\pi}{\gamma_0}\tau\right)\right), \tag{6}$$

in which ϕ denotes a designed modulation function and $\gamma_0 > 0$. The FO SM-ESC is depicted in Fig. 1.

3. Stability analysis of FO sliding mode based extremum seeking control

3.1. Stability analysis of the FO SM-ESC

First, one has $\dot{\tau} = \gamma_1 F' v + \xi_s$, where $\xi_s := -\gamma_1 k_r + \gamma_{20}D_t^q\mathrm{sgn}(e)$. Denoting $k = \gamma_1 F'$, one has $\dot{\tau} = k(v + \omega)$, in which $\omega := (-\gamma_1 k_r + \gamma_{20}D_t^q\mathrm{sgn}(e))/k$. From Assumption 3, there exists $0 < \tilde{k} \le \gamma_1\varepsilon$ such that $\tilde{k} \le |k|$, $\forall\theta \notin D_\delta$. ϕ will be designed such that $\phi\tanh^2(\sin(\pi\tau/\gamma_0)) \ge |\omega| + \alpha$, in which $\alpha \ge 0$. Hence, $|\omega| \le \hat{\omega}$, where $\hat{\omega} \le (\gamma_1 k_r + \gamma_2\varpi)/\tilde{k}$. Next, one possible ϕ will be proposed to guarantee the control performance.

Theorem 1. *Consider the system (1)–(2) with the FO SM-ESC (6). Outside the* D_δ*, if* ϕ *in (6) satisfies*

$$\phi\tanh^2(\sin(\pi\tau/\gamma_0)) := [(\gamma_1 k_r + \gamma_2\varpi) + \|y_t\|e^{-\alpha_1 t}]/\tilde{k} + \alpha, \tag{7}$$

where $\alpha_1 > 0$*, then, while* $\theta \notin D_\delta$*, one has* (i)*: no finite-time escape happens in the closed-loop signals* ($t_M \to +\infty$) *and* (ii) *a sliding mode on* $\tau(t) = l\gamma_0$ *is arrived in finite time for some integer* l*.*

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Proof. See Appendix B.

Next, it will be shown that the FO SM-ESC with ${}_0D_t^q\mathrm{sgn}(e)$, $0 \le q < 1$ and ϕ ensures that θ arrives at D_δ of the unknown maximizer θ^* defined in Assumption 3.

Theorem 2. *Consider the system (1)–(2) under the controller (6) with the function (7). If Assumptions 1–3 are satisfied, then:* (i) *D_δ in Assumption 3 is globally attractive that is achieved in finite time and* (ii) *for small enough ε, the oscillations around y^* of y can be made of order $O(\gamma_0)$. Furthermore, all closed-loop system signals are uniformly bounded (UB) except for $\tau(t)$ that is an argument of an FO sign function in (5).*

Proof. See Appendix C.

3.2. Comparison between IO SM-ESC and FO SM-ESC

In our previous work Yin et al. (2012), the IO SM-ESC with $\tanh(\sin(\pi\tau/\gamma_0))$ was developed. τ in the IO SM-ESC is given by

$$\tau = \gamma_1 e + \gamma_2 \mathrm{sgn}(e), \tag{8}$$

where $e = y - y_r$. θ is proposed to satisfy $\dot\theta = v$, in which $v = \phi\tanh(\sin(\pi\tau/\gamma_0))$. The IO SM-ESC can ensure that y approaches y^* and stays in it thereafter. Since ${}_0D_t^0\mathrm{sgn}(e) = \frac{1}{\Gamma(1-0)}\frac{d}{dt}\int_0^t \frac{\mathrm{sgn}(e(\epsilon))}{(t-\epsilon)^0}d\epsilon = \mathrm{sgn}(e)$, the FO SM-ESC with ${}_0D_t^q\mathrm{sgn}(e)$, $0 \le q < 1$ is an FO/IO controller synthesis (*IO controller with* $\mathrm{sgn}(e)$ *and FO controller with* ${}_0D_t^q\mathrm{sgn}(e)$, $0 < q < 1$). In what follows, a comparison and analysis for the FO/IO SM-ESC synthesis reveal the reason the FO SM-ESC can have a better tracking performance.

One property of the good tracking performance is that y can quickly reach y^*. In order to show how to obtain a faster tracking performance under the FO SM-ESC, the following remark is presented.

Remark 2. The proof of Theorem 1 uses $\tanh(\sin(\pi\tau/\gamma_0))$ to ensure the existence of a series of sliding surfaces $s_l = \tau - l\gamma_0$ and realization of τ-sliding modes, while $\theta \notin D_\delta$. One has $\dot e = -(\gamma_2/\gamma_1){}_0D_t^q\mathrm{sgn}(e)$ from $\dot\tau = 0$. From Lemma 2, one has the following relationship (in which $y(0), y_r(0)$ are the initial conditions of y and y_r)

(I) When $e(0) = y(0) - y_r(0) < 0$ (i.e.$y(0) < y_r(0)$),
one has $e \nearrow$ since $\dot e = -(\gamma_2/\gamma_1){}_0D_t^q\mathrm{sgn}(e) > 0$,
then, y will be close to y_r.

(II) When $e(0) = y(0) - y_r(0) < 0$ (i.e. $y(0) < y_r(0)$),
one has $e \searrow$ since $\dot e = -(\gamma_2/\gamma_1){}_0D_t^q\mathrm{sgn}(e) < 0$,
then, y will be close to y_r.

Hence, outside a small γ_0-neighborhood of y^*, y tries to track y_r. One can assure that y approaches y^*. Moreover, it is more likely to choose a series of q, $(0 < q < 1)$ such that $|{}_0D_t^q\mathrm{sgn}(e)|$ is bigger than 1 (i.e., $|{}_0D_t^q\mathrm{sgn}(e)| \gg 1 = |\mathrm{sgn}(e)|$) during the initial time interval. This is the second important property of ${}_0D_t^q\mathrm{sgn}(e)$ where y determined by the FO SM-ESC will create a stronger push to any arbitrary small vicinity of y^* during this time interval, even if $|{}_0D_t^q\mathrm{sgn}(e)| \le 1$ during the next time interval. The simulation and experimental results in Section 5 can help to demonstrate the advantages of the FO SM-ESC.

Remark 3. Especially, the maximum point of some systems (e.g. MPPT of the PV plant Malek et al., 2012) may be constantly changing with environmental variations, but the rough upper bound of the maximum point is known. y_{r0} can be chosen such that y_{r0} is bigger than the upper bound, such that y_r can draw y close to y^* and $e < 0$. Thus, y with the positive rate $\dot y = k_r - (\gamma_2/\gamma_1){}_0D_t^q\mathrm{sgn}(e)$

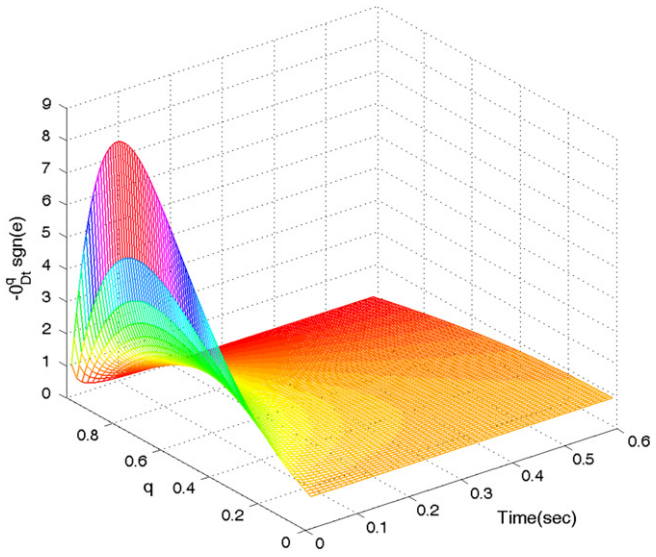

Fig. 2. Relationship between ${}_0D_t^q\mathrm{sgn}(e)$ and q, for $e(t) < 0$, $0 \le q < 1$.

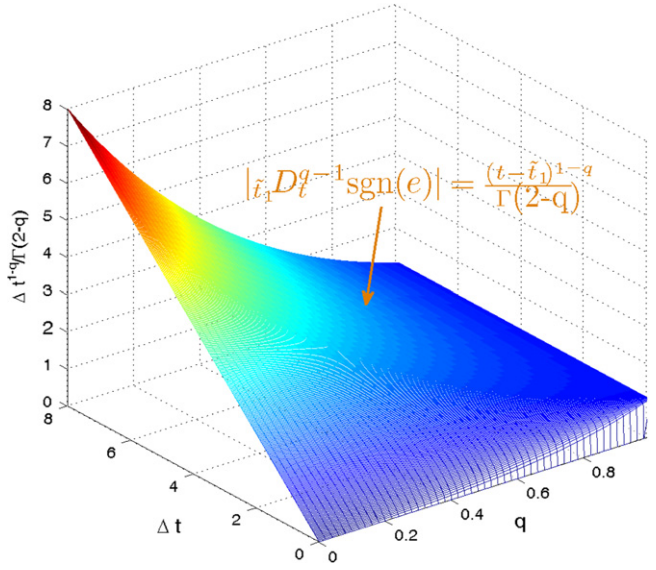

Fig. 3. Relationship between $\Delta t^{1-q}/\Gamma(2-q)$ and q, for $e(t) < 0$, $0 \le q < 1$.

approaches y^* when y is out of the γ_0 vicinity of y^*. Similarly, the rate of y under the IO SM-ESC is $k_r + \gamma_2/\gamma_1$. As shown in Fig. 2, $-{}_0D_t^q\mathrm{sgn}(e) \gg 1 = -\mathrm{sgn}(e)$, $0 < q < 1$ during the initial time interval. Thus, $k_r - (\gamma_2/\gamma_1){}_0D_t^q\mathrm{sgn}(e) > k_r + \gamma_2/\gamma_1$ in the time interval. One can assure that the FO SM-ESC with $0 < q < 1$ has a faster tracking performance than that the IO SM-ESC does. The MPPT of the PV model in Section 4 shows the advantages of the FO SM-ESC.

Another property of the good tracking performance is to drive towards a smaller neighborhood of y^*. The following remark illustrates the reason the oscillations under the FO SM-ESC can be reduced.

Remark 4. In Theorem 2, the oscillations around y^* of y can be made of order $O(\gamma_0)$. Now, we consider ${}_{\tilde t_1}D_t^{q-1}\mathrm{sgn}(e)$ in (C.1). Since $e(t) < 0$, $\forall t > \tilde t_1$, one has $|{}_{\tilde t_1}D_t^{q-1}\mathrm{sgn}(e)| = \frac{(t-\tilde t_1)^{1-q}}{\Gamma(2-q)}$. From (C.1), one has

$$|\tilde y(t)| \le \gamma_1|\tilde\tau(t)| + \gamma_1 k_r(t - \tilde t_1) + \frac{\gamma_2}{\Gamma(2-q)}(t-\tilde t_1)^{1-q},$$

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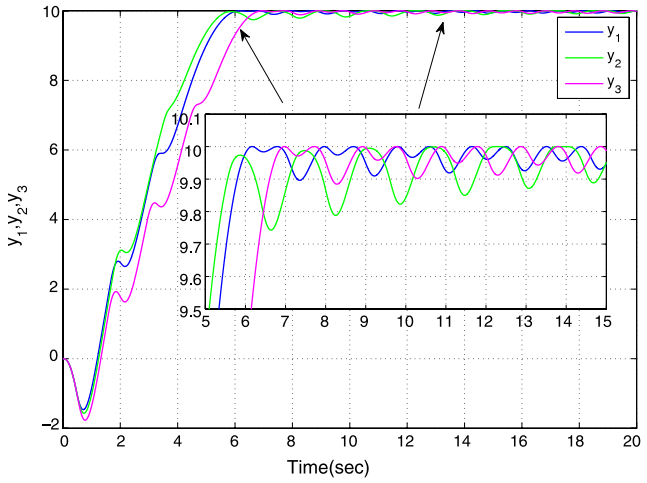

Fig. 4. Time responses of y_1 (blue line) under the FO SM-ESC, y_2 (green line) under the IO SM-ESC in Oliveira et al. (2011) and y_3 (purple line) under the IO SM-ESC in Pan et al. (2003). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$\forall t > \tilde{t}_1$. Similarly, one has $|\tilde{y}(t)| \leq \gamma_1|\tilde{\tau}(t)| + \gamma_1 k_r(t-\tilde{t}_1) + \gamma_2(t-\tilde{t}_1),\ \forall t > \tilde{t}_1$ by using the IO SM-ESC. Since $t-\tilde{t}_1 = \frac{(t-\tilde{t}_1)}{\Gamma(2)}$, it is more likely for the FO SM-ESC to obtain a smaller vicinity of y^*. Denoting $\Delta t = t - \tilde{t}_1,\ t > \tilde{t}_1$, the changes in $-\frac{\Delta t^{1-q}}{\Gamma(2-q)}$, with different q and Δt, in which $q \in [0, 0.99]$, $\Delta t \in [0, 8]$ are depicted in Fig. 3. Since there exist many q, t such that $0 < \frac{1}{\Gamma(2-q)}(t-\tilde{t}_1)^{1-q} < t - \tilde{t}_1$, the oscillations around y^* of y under the FO SM-ESC can be made smaller.

4. Simulation and experimental results

4.1. Simulation results

In this subsection, two examples are used to illustrate the effectiveness of the proposed FO SM-ESC scheme.

Example 1. Consider the double-integrator system (Pan et al., 2003)

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \tag{9}$$

with $u(t) = -30(x_1(t) - \theta(t)) - 11x_2(t)$ and the output $y(t) = -10(x_1(t) - 5)^2 + 10$. $y(t)$ reaches its maximum $y^* = 10$ at $x_1(t) = 5$. $x_1(0) = \theta(0) = 4, x_2(0) = y_r(0) = 0$ and $k_r = \gamma_0 = \gamma_1 = 2, \phi = 0.25$ are chosen, the same as in Pan et al. (2003). $\gamma_1 = 0.5$ is selected in the simulation. Fig. 4 shows the comparison of y_1 under the FO SM-ESC with $q = 0.4$, y_2 under the IO SM-ESC law in Oliveira, Hsu, and Peixoto (2011) and y_3 under the IO SM-ESC in Pan et al. (2003). It shows that y_1 has faster tracking performance and higher accuracy than y_2, y_3.

Next, the simulations are performed while q is changed. Fig. 5 shows that y_1, y_3 under the FO SM-ESC with $q_1 = 0.5, q_3 = 0.2$ arrive at y^* more rapidly than y_2 under the IO SM-ESC, since $|{}_0D_t^{0.5}\mathrm{sgn}(e)| \gg 1$, $|{}_0D_t^{0.3}\mathrm{sgn}(e)| \gg 1$ during $[0, 0.4]$. Furthermore, y_1 and y_3 achieve to a smaller vicinity of y^* than y_2 does. Thus, the FO SM-ESC can have better tracking performance.

Example 2. Consider the MPPT problem of a PV array. In the PV system, the net current of the cell is the difference of the photocurrent I_L and the normal diode current I_0 : $I = I_L - I_0\left(e^{\frac{q(V+IR_S)}{nKT}} - 1\right)$, that I, V are the current and voltage, n denotes a variable parameter (instead of being fixed at either 1 or 2). We consider the model in Malek et al. (2012) and Walker (2001). The PV module provides about 48 W of nominal maximum power, when $T = 25°$ and $G = 1000\ \mathrm{W/m^2}$. Then, we describe the design of the proposed MPP. Consider the following auxiliary first order nonlinear system:

$$\dot{V} = v, \qquad I = h(V) = I_L - I_0\left(e^{\frac{q(V+IR_S)}{nKT}} - 1\right). \tag{10}$$

To maximize the power output $P = VI$, we employ the FO SM-ESC (6) with $q = 0.1$ for (10). Matlab s-function is used to express the PV model, adopted from Walker (2001). $\phi = (\gamma_1 + \gamma_2 + \gamma_1 k_r)/\varepsilon + \alpha$ and $\varepsilon = 0.1\gamma_0$ are used. Let $V(0) = 0, y_r(0) = 50$ and $\gamma_0 = 0.01, \gamma_1 = 0.5, \gamma_2 = 0.1, k_r = 0.5$ and $\alpha = 0.1$ in our simulations. Fig. 6 shows the comparison between P_1 under the FO SM-ESC and P_2 under the IO SM-ESC. It shows that P_1 converges to the MPP faster than P_2. Moreover, the amplitude of the oscillations around P_{MPP} of P can be reduced by utilizing the FO SM-ESC.

Next, the simulations are done while q is changed. Since $|{}_0D_t^{0.17}\mathrm{sgn}(e)|$, $|{}_0D_t^{0.25}\mathrm{sgn}(e)|$, $|{}_0D_t^{0.4}\mathrm{sgn}(e)|$ are bigger than 1 during $[0, 0.2]$, $P_{FO-SM-ESC}$ reaches P^* than $P_{IO-SM-ESC}$, as shown in Fig. 7. They show the advantage of the FO SM-ESC.

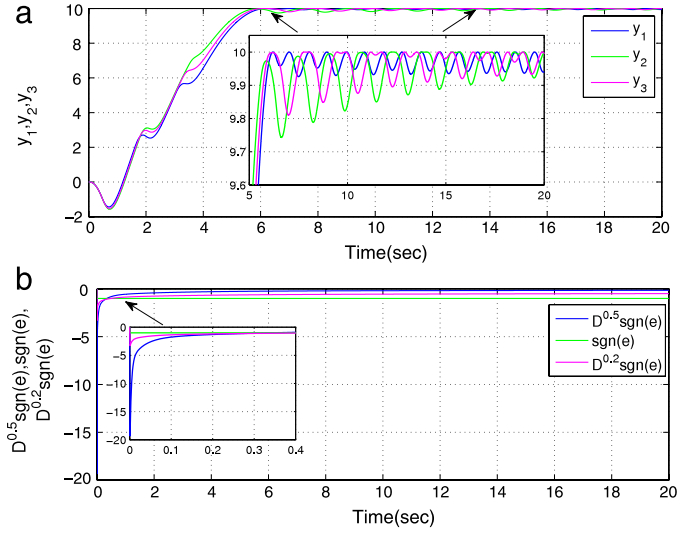

Fig. 5. (a) Time responses of y_1 (blue line) and y_3 (purple line) under the FO SM-ESC with $q_1 = 0.5, q_3 = 0.3$ and y_2 (green line) under the IO SM-ESC in Oliveira et al. (2011); (b) time responses of $D^{0.5}\mathrm{sgn}(e)$, $D^{0.3}\mathrm{sgn}(e)$ and $\mathrm{sgn}(e)$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

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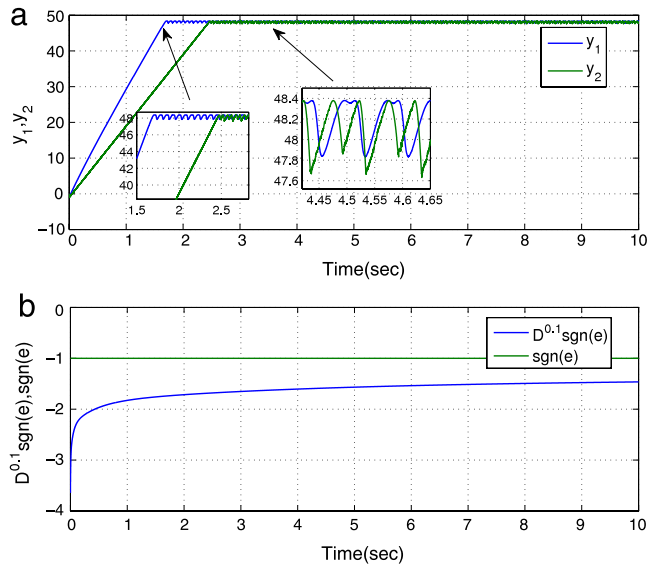

Fig. 6. (a) Time responses of P_1 (blue line) under our control and P_2 (green line) under the IO SM-ESC law in Oliveira et al. (2011); (b) time responses of ${}_0D_t^{0.1}\text{sgn}(e)$ and sgn(e). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

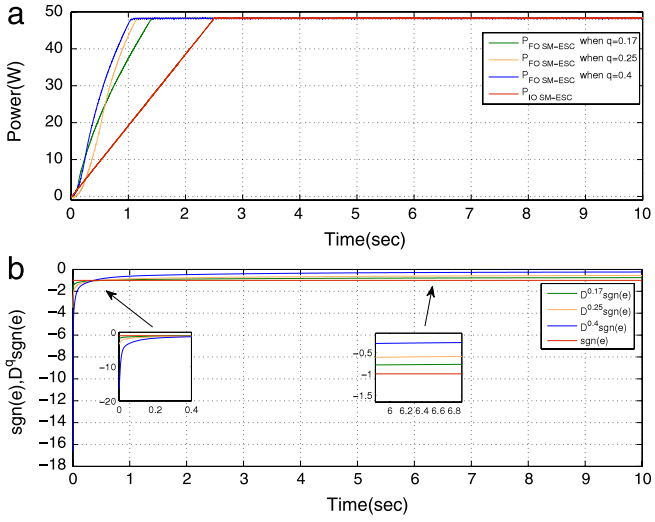

Fig. 7. (a) Time responses of P_i, $(i = 1, 2, 3, 4)$ for different fractional orders; (b) time responses of ${}_0D_t^{0.17}\text{sgn}(e)$, ${}_0D_t^{0.25}\text{sgn}(e)$, ${}_0D_t^{0.4}\text{sgn}(e)$ and sgn(e).

4.2. Experimental results

A fractional horsepower dynamometer has been applied to model the PV model in Malek et al. (2012). In Fig. 8, the dynamometer (Tarte, Chen, Ren, & Moore, 2006) has a DC motor, an optical encoder, a hysteresis brake, a load cell, and a tachometer. It communicates with a Quanser Q4 terminal board for the connect with the Matlab/Simulink Real-Time Workshop environment by WinCon 4.0. The DC motor is identified

$$G_m(s) = \frac{1.52}{1.01s + 1}. \tag{11}$$

In Malek et al. (2012), the PV panel has been built by the hysteresis brake. The break output torque is considered as output current of PV modules. Thus, the power is a product of the angular velocity and the current of PV modules. The proposed scheme is given in Fig. 9. The brake and motor are separately driven by an advanced motion controls brush type PWM servo amplifier Model 50A8. These controllers obtain analog signals from data acquisition

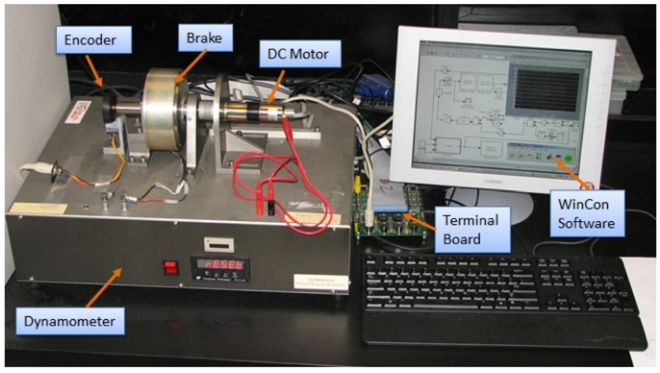

Fig. 8. The fractional horse power dynamometer.

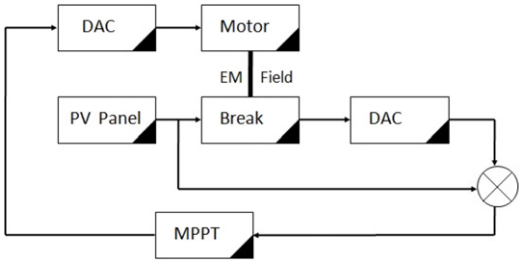

Fig. 9. Modeling the PV panel using the fractional horse power dynamometer.

hardware. The PWM controllers use the signals to set the voltage output to the motor or the brake.

In this experiment, the FO SM-ESC is applied to the dynamometer as the PV model. The Simulink model used for the experiment is shown in Fig. 10. The comparison between the FO/IO SM-ESC for PV model is depicted in Figs. 11-12. Fig. 11(a) shows the comparison of P_i, $(i = 0, 1, 2, 3, 4)$ under the FO SM-ESC with $q_0 = 0, q_1 = 0.1, q_2 = 0.2, q_3 = 0.35, q_4 = 0.42$. Fig. 11(b) shows the comparison between sgn(e) and ${}_0D_t^{q_i}\text{sgn}(e)$ with q_i, $(i = 1, 2, 3, 4)$. As shown in Fig. 11, since $|{}_0D_t^{q_i}\text{sgn}(e)| \gg 1$, $i = 1, 2, 3, 4$ during $[0, 0.05]$, P_i, $(i = 1, 2, 3, 4)$ with the faster convergence speed achieve to the γ_0-neighborhood of P^* when compared to P_0. The corresponding V_i and I_i, $(i = 0, 1, 2, 3, 4)$ are depicted in Fig. 12(a) and (b), respectively. They show that the FO SM-ESC has a better tracking performance.

5. Conclusion

In this paper, the FO SM-ESC law with ${}_0D_t^q\text{sgn}(e)$, $0 \le q < 1$ was proposed for the optimization of nonlinear systems. The FO SM-ESC was developed by combining the FO sign function and the tanh periodic switching function. It has been shown that the FO sign function can help to own faster convergence to an arbitrary smaller vicinity of the optimal point, when compared to the sign function. Simulation and experimental results show the advantages of the proposed FO control scheme.

Appendix A. Proof of Lemma 2

For $\forall t > 0$, there exists a time series $\{t_i\}$ in which $t_0 = 0, t_{j_1} > t_{j_2}$ (when $j_1 > j_2$, $\forall j_1, j_2 \in N$) such that $t \in (t_k, t_{k+1}], \exists k \in N$ and $e(t') \le 0$, $\forall t' \in (t_i, t_{i+1}]$, if $e(t') \ge 0$, $\forall t' \in (t_{i-1}, t_i]$ when $i < k$. Moreover, we require $e(\bar{t}) \ne 0$, where $\bar{t}$ is the first time in (t_i, t_{i+1}). Thus, the details can be divided into two cases: (a) while $e(t') \ge 0$, $\forall t' \in (t_0, t_1]$, $e(t') \le 0$, $\forall t' \in (t_1, t_2]$, and then, $e(t') \ge 0$, $\forall t' \in (t_2, t_3]$, and so on. (b) While $e(t') \le 0$, $\forall t' \in (t_0, t_1]$, $e(t') \ge 0$, $\forall t' \in (t_1, t_2]$, and then, $e(t') \le 0$, $\forall t' \in (t_2, t_3]$, and so on.

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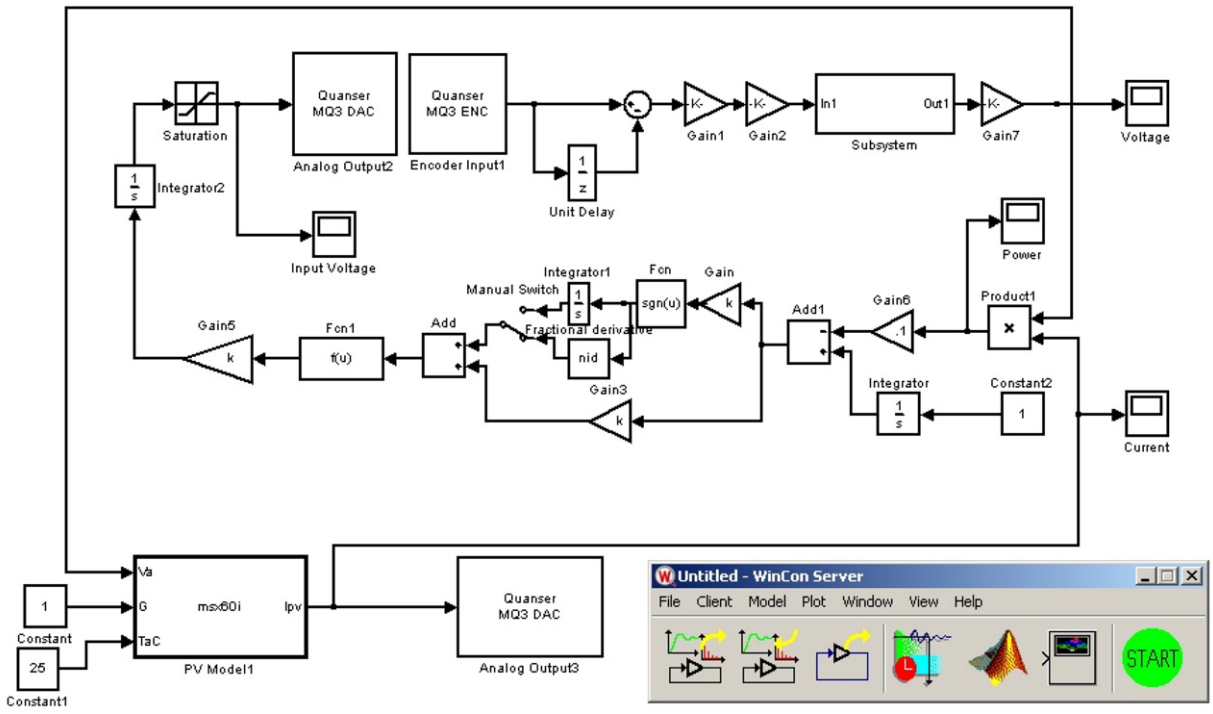

Fig. 10. Simulink model used in the FO SM-ESC real time experiment using RTW Windows Target.

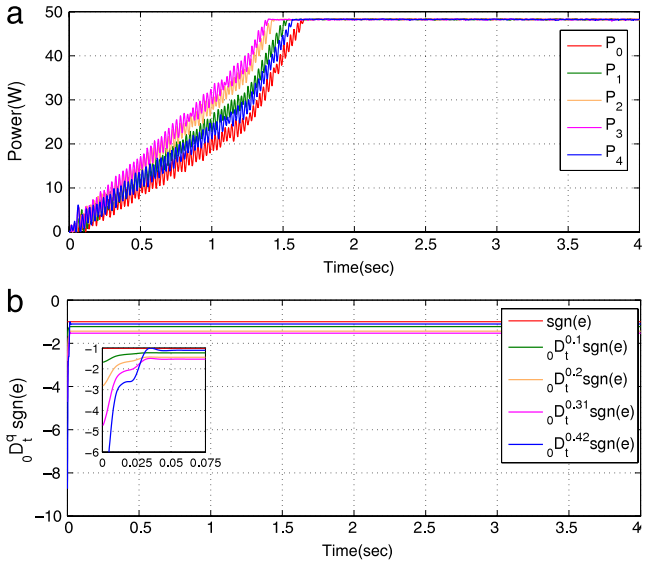

Fig. 11. (a) $P_i,\ i = 1, 2, 3, 4$ under the FO SM-ESC and P_0 under the IO SM-ESC; (b) time histories of ${}_0D_t^{0.1}\mathrm{sgn}(e)$, ${}_0D_t^{0.2}\mathrm{sgn}(e)$, ${}_0D_t^{0.31}\mathrm{sgn}(e)$, ${}_0D_t^{0.42}\mathrm{sgn}(e)$ and $\mathrm{sgn}(e)$.

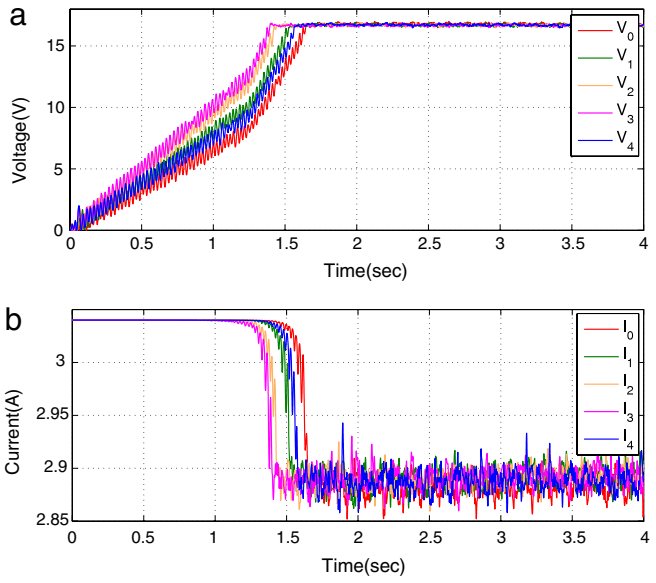

Fig. 12. (a) The corresponding voltages $V_i,\ (i = 0, 1, 2, 3, 4)$; (b) the corresponding currents $I_i,\ (i = 0, 1, 2, 3, 4)$.

According to the integral properties, one has

$$ {}_0D_t^q\mathrm{sgn}(e(t)) = \frac{[f_0(t) + f_1(t) + \cdots + f_k(t)]}{\Gamma(1-q)}, \tag{A.1}$$

where $f_i(t) = \frac{d}{dt}\int_{t_i}^{t_{i+1}} \frac{\mathrm{sgn}(e(\tau))}{(t-\tau)^q}d\tau,\ (i = 0, 1, 2, \ldots, k-1), f_k(t) = \frac{d}{dt}\int_{t_k}^{t} \frac{\mathrm{sgn}(e(\tau))}{(t-\tau)^q}d\tau$.

First of all, we consider the first case $e(t) > 0$. Hence, $e(t') \geq 0,\ \forall t' \in (t_k, t_{k+1}]$ from the above analysis. Let $t_{k0} = t_k, t_{kl_k} = t$, one can obtain $(t_k, t] = (t_{k0}, t_{k1}] \cup (t_{k1}, t_{k2}] \cup \cdots \cup (t_{kl_k-1}, t_{kl_k}]$, where if $e(t') \equiv 0,\ \forall t' \in (t_{ki}, t_{ki+1}], e(t') \geq 0,\ \forall t' \in (t_{ki+1}, t_{ki+2}]$ in which $e(t') = 0$ just happens at an isolate point t'. Since $e(t) > 0, e(t') \geq 0,\ \forall t' \in (t_{kl_k-1}, t_{kl_k}]$. Thus, one has

$$\frac{d}{dt}\int_{t_{kl_k-1}}^{t} \frac{\mathrm{sgn}(e(\tau))}{(t-\tau)^q}d\tau = (t - t_{kl_k-1})^{-q},$$

$$\frac{d}{dt}\int_{t_{kl_k-2}}^{t_{kl_k-1}} \frac{\mathrm{sgn}(e(\tau))}{(t-\tau)^q}d\tau = \frac{d}{dt}\int_{t_{kl_k-2}}^{t_{kl_k-1}} \frac{0}{(t-\tau)^q}d\tau = 0,$$

$$\frac{d}{dt}\int_{t_{kl_k-3}}^{t_{kl_k-2}} \frac{\mathrm{sgn}(e(\tau))}{(t-\tau)^q}d\tau = \frac{d}{dt}\int_{t_{kl_k-3}}^{t_{kl_k-2}} \frac{1}{(t-\tau)^q}d\tau = (t - t_{kl_k-3})^{-q} - (t - t_{kl_k-2})^{-q},$$

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and so on, one can conclude

$$f_k(t) = (t-t_{k0})^{-q} - (t-t_{k1})^{-q} + (t-t_{k2})^{-q} - (t-t_{k3})^{-q} + \cdots + (t-t_{kl_k-1})^{-q} - 0. \quad (A.2)$$

Since $(t-t_{ki})^{-q}$ is an increasing function in t_{ki}, $f_k(t) \geq (t-t_{k0})^{-q} = (t-t_k)^{-q}$. Then, we will discuss $f_i(t)$. When $e(t') \geq 0$, $\forall t' \in (t_i, t_{i+1}]$, (let $(t_i, t_{i+1}] = (t_{i0}, t_{i1}] \cup (t_{i1}, t_{i2}] \cup \cdots \cup (t_{il_i-1}, t_{il_i}]$ in which $t_{i0} = t_i$, $t_{il_i} = t_{i+1}$), $f_i(t)$ can be calculated as follows:

$$\begin{aligned} f_i(t) &= (t-t_{i0})^{-q} - (t-t_{i1})^{-q} + (t-t_{i2})^{-q} - (t-t_{i3})^{-q} \\ &\quad + \cdots + (t-t_{il_i-1})^{-q} - (t-t_{il_i})^{-q} \\ &\geq (t-t_i)^{-q} - (t-t_{i+1})^{-q}, \end{aligned}$$

or

$$\begin{aligned} f_i(t) &= (t-t_{i0})^{-q} - (t-t_{i1})^{-q} + (t-t_{i2})^{-q} - (t-t_{i3})^{-q} \\ &\quad + \cdots + (t-t_{il_i-2})^{-q} - (t-t_{il_i-1})^{-q} \\ &> (t-t_i)^{-q} - (t-t_{i+1})^{-q}. \end{aligned} \quad (A.3)$$

When $e(t') \leq 0$, $\forall t' \in (t_i, t_{i+1}]$, let $(t_i, t_{i+1}] = (t_{i0}, t_{i1}] \cup (t_{i1}, t_{i2}] \cup \cdots \cup (t_{il_i-1}, t_{il_i}]$ in which $t_{i0} = t_i$, $t_{il_i} = t_{i+1}$. If $e(t') \equiv 0$, $\forall t' \in (t_{ij}, t_{ij+1}]$, $e(t') \leq 0$, $\forall t' \in (t_{ij+1}, t_{ij+2}]$ in which $e(t') = 0$ just happens at an isolate point t'. Based on the divided condition for $(t_i, t_{i+1}]$, one has $e(t') \leq 0$, $\forall t' \in (t_{i0}, t_{i1}]$. Hence, one can conclude that

$$\begin{aligned} \frac{d}{dt}\int_{t_{i0}}^{t_{i1}} \frac{\mathrm{sgn}(e(\tau))}{(t-\tau)^q} d\tau &= (t-t_{i1})^{-q} - (t-t_{i0})^{-q}, \\ \frac{d}{dt}\int_{t_{i1}}^{t_{i2}} \frac{\mathrm{sgn}(e(\tau))}{(t-\tau)^q} d\tau &= \frac{d}{dt}\int_{t_{i1}}^{t_{i2}} \frac{0}{(t-\tau)^q} d\tau = 0, \end{aligned} \quad (A.4)$$

and so on, so one has

$$f_i(t) = (t-t_{i1})^{-q} - (t-t_{i0})^{-q} + (t-t_{i3})^{-q} - (t-t_{i2})^{-q} + \cdots + (t-t_{il_i})^{-q} - (t-t_{il_i-1})^{-q} > 0,$$

or

$$f_i(t) = (t-t_{i1})^{-q} - (t-t_{i0})^{-q} + (t-t_{i3})^{-q} - (t-t_{i2})^{-q} + \cdots + (t-t_{il_i-1})^{-q} - (t-t_{il_i-2})^{-q} > 0. \quad (A.5)$$

From the above analysis, it is clear that

$$\sum_{i=0}^{k} f_i(t) > (t-t_k)^{-q} + (t-t_{k-2})^{-q} - (t-t_{k-1})^{-q} + \cdots + t^{-q} - (t-t_1)^{-q} > 0,$$

or

$$\sum_{i=0}^{k} f_i(t) > (t-t_k)^{-q} + (t-t_{k-2})^{-q} - (t-t_{k-1})^{-q} + \cdots + (t-t_1)^{-q} - (t-t_2)^{-q} > 0. \quad (A.6)$$

Thus, we have

$${}_0D_t^q \mathrm{sgn}(e(t)) = \frac{[f_0(t) + f_1(t) + \cdots + f_k(t)]}{\Gamma(1-q)} > 0. \quad (A.7)$$

Next, we consider the second case when $e(t) < 0$. Similar to the first case, we have

$${}_0D_t^q \mathrm{sgn}(e(t)) = \frac{[f_0(t) + f_1(t) + \cdots + f_k(t)]}{\Gamma(1-q)} < 0. \quad (A.8)$$

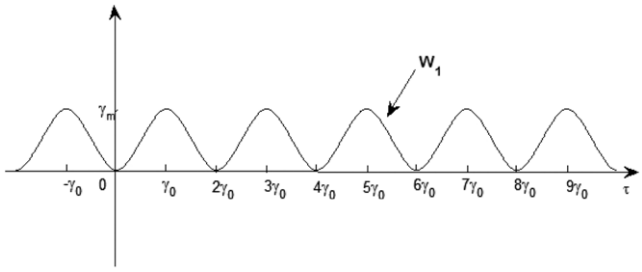

Fig. B.1. W_1 when $k < 0$.

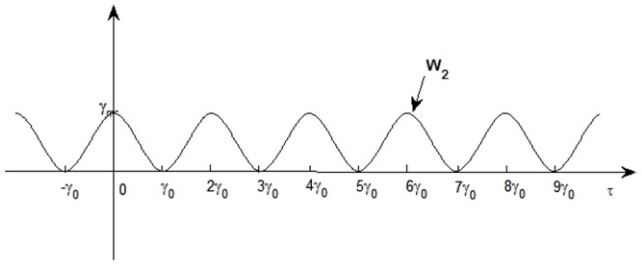

Fig. B.2. W_2 when $k > 0$.

Appendix B. Proof of Theorem 1

Consider the non-negative functions, see Figs. B.1–B.2

$$W_1(\tau) = \int_0^{\tau} \tanh\left(\sin\left(\frac{\pi}{\gamma_0}\epsilon\right)\right) d\epsilon, \qquad W_2(\tau) = \gamma_m - W_1,$$

where γ_m denotes the maximum value of $W_1(\tau)$. One can obtain $\dot{W}_i \leq k\{\phi \tanh^2[\sin(\pi\tau/\gamma_0)]\} + |k||\omega|$, $i = 1, 2$, since $|\tanh(\sin(\pi\tau/\gamma_0))| \leq 1$. According to $|k||\omega| \leq |k|\hat{\omega}$ and $-|k| \leq \tilde{k}$, one can conclude that

$$\dot{W}_1 \leq -\|y_t\| e^{-\alpha_1 t} - \tilde{k}\alpha, \quad \text{if } \mathrm{sgn}(k) < 0, \quad (B.1)$$

$$\dot{W}_2 \leq -\|y_t\| e^{-\alpha_1 t} - \tilde{k}\alpha, \quad \text{if } \mathrm{sgn}(k) > 0, \quad (B.2)$$

are satisfied nearly everywhere, with $\alpha \geq 0, \alpha_1 > 0$. Next, the property (i) of Theorem 1 will be demonstrated.

(i) Assume that $|\tau(t)|$ escapes in some finite time $t_1 \in [0, t_M)$. Then, $e(t)$ and $y(t)$ escape at $t = t_1$. Hence, $\exists t_2 \in [0, t_1)$ such that $\|y_t\| \geq e^{\alpha_1 t}[\alpha_2 - \tilde{k}\alpha]$, $\forall t \in [t_2, t_1)$, where $\alpha_2 \geq 0$. Thus, from (B.1)–(B.2), one has $\dot{W}_1 \leq -\alpha_2$ or $\dot{W}_2 \leq -\alpha_2$, $\forall t \in [t_2, t_1)$, independently of $\mathrm{sgn}(k)$. Because $\tau(t)$ escapes at $t = t_1$ and is continuous, $\exists t_e \in [t_2, t_1)$ and an integer l_τ such that $\tau(t_e) = l_\tau\gamma_0$. Hence, $W_1(t_e) = 0$ (even number l_τ) or $W_2(t_e) = 0$ (odd number l_τ). Obviously, $W_1(t) = 0$ or $W_2(t) = 0$, $\forall t \in [t_e, t_1)$, see Figs. B.1–B.2. Note that $W_i(\tau(t))$ was changed by $W_i(t)$, $i = 1, 2$. Hence, $\tau(t) = l_\tau\gamma_0$, $\forall t \in [t_e, t_1)$ is uniformly bounded, i.e., a contradiction. Thus, τ, e and y cannot escape in finite time. Furthermore, θ and all closed loop signals cannot escape in $(t_M \to +\infty)$.

(ii) From (i), $\exists \bar{t} \geq 0$ such that $\dot{W}_i \leq -\alpha_{\bar{t}}$, $\forall t \geq \bar{t}$ and $0 < \alpha_{\bar{t}} < \tilde{k}\alpha$, for $i = 1$ or 2. Hence, $W_i(t) \leq -\alpha_{\bar{t}}(t-\bar{t}) + W_i(\bar{t})$, $\forall t \geq \bar{t}$. Thus, $\exists t^* \geq \bar{t}$ such that $W_i(t) = 0$, $\forall t \geq t^*$. Hence, $\tau = l\gamma_0$ for that $W_1(\tau) = 0 (W_2(\tau) = 0)$ just happens at even(odd) number l. In the small vicinity around $\tau = l\gamma_0$, $\mathrm{sgn}(\tanh(\sin(\pi\tau/\gamma_0))) = \mathrm{sgn}(\tanh(\tau - l\gamma_0))$, for even number l, or $\mathrm{sgn}(\tanh(\sin(\pi\tau/\gamma_0))) = \mathrm{sgn}(-\tanh(\tau - l\gamma_0))$, for odd number l. Selecting $V = 0.5(\tau - l\gamma_0)^2$, one has $\dot{V} = (\tau - l\gamma_0)(k(\phi \tanh\left(\sin\left(\frac{\pi}{\gamma_0}\tau\right)\right) + \omega)) \leq 0$. Therefore, a sliding mode happens in finite time on one of the surfaces $\tau = l\gamma_0$, independently of $\mathrm{sgn}(k)$.

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Appendix C. Proof of Theorem 2

(i) Attractiveness of D_δ: Suppose that $\theta \notin D_\delta$, $\forall t \in [0, t_M)$. From Theorem 1, $\exists t_f$ such that $\dot{\tau} = 0$. Hence, $\dot{e} = -(\gamma_2/\gamma_1)_0 D_t^q \mathrm{sgn}(e)$, $\forall t \geq t_f$. From Lemma 2, y_r draws y close in y^*. Since $y \leq y^*$ and y_r strictly increases with time, $y_r > y^* \geq y$ and $e < 0$ for large enough t. So y increases with $\dot{y} = k_r - (\gamma_2/\gamma_1)_0 D_t^q \mathrm{sgn}(e) > 0$ for large enough t, i.e. $y = F(\theta)$ must be close to y^*. So, θ approaches D_δ, which is a contradiction. Thus, D_δ is accomplished in finite time. Thus, θ remains or oscillates around D_δ. Hence, y stays in or oscillates around some small vicinity of y^*. These oscillations appear due to the recurrent changes in the sign of F' at the vicinity of (θ^*, y^*) where $F' = 0$. Meantime, τ can switch from one sliding surface $\tau = l\gamma_0$ (odd number l ($\mathrm{sgn}(k) > 0$)) to another surface (even number l ($\mathrm{sgn}(k) < 0$)). Since y could start oscillations around y^* with increasing maximum amplitude, it should prove that $|y - y^*|$ can be made ultimately of order $O(\gamma_0)$.

(ii) Oscillations of order $O(\gamma_0)$ around y^*: From Assumption 3, if θ stays in D_δ, $\forall t$, one has $|y - y^*| = O(\gamma_0)$. Next, it will illustrate that $|y - y^*| = O(\gamma_0)$ when θ oscillates around D_δ. One has $y_r > y^* \geq y$, $\forall t$ large enough. Thus, $\exists \tilde{t} > 0$ such that $e(t) < 0$, $\forall t > \tilde{t}$. From Lemma 2, ${}_0D_t^q \mathrm{sgn}(e) < 0$, $\forall t > \tilde{t}$. One has $\tau(t) = \gamma_1 y(t) - \gamma_1 y_r(t) + \gamma_2({}_0D_t^{q-1}\mathrm{sgn}(e))$. Suppose that θ arrives at D_δ from inside at $\tilde{t}_1 > \tilde{t}$ and $\tau(t)$ is not in sliding surface when $t = \tilde{t}_1$. D_δ is constant when $\tau(t)$ is in sliding manifold. Define $\tilde{\tau}(t) = \tau(t) - \tau(\tilde{t}_1)$, $\tilde{y}(t) = y(t) - y(\tilde{t}_1)$, one has

$$\tilde{\tau}(t) = \gamma_1 \tilde{y}(t) - \gamma_1 k_r (t - \tilde{t}_1) + \gamma_2({}_{\tilde{t}_1}D_t^{q-1}\mathrm{sgn}(e)). \tag{C.1}$$

From Lemma 1, $\exists K > 0$ such that $|{}_{\tilde{t}_1}D_t^{q-1}\mathrm{sgn}(e)| < K|\mathrm{sgn}(e)| = K$. Hence, one has $|\tilde{y}(t)| \leq \gamma_1|\tilde{\tau}(t)| + \gamma_1 k_r(t - \tilde{t}_1) + K$. Let $\tilde{t}_2$ be the first time when $\tau(t)$ arrives at the next sliding surface $\tau(t) = \tau(\tilde{t}_2)$ and $\tilde{t}_3$ is the first time when θ reaches the frontier of D_δ again (from outside). One has $\tilde{t}_2 \geq \tilde{t}_1$, $\tilde{t}_3 \geq \tilde{t}_1$. Then, consider two cases: (I) $\tilde{t}_3 \leq \tilde{t}_2$ and (II) $\tilde{t}_3 > \tilde{t}_2$.

For (I), let $t \in [\tilde{t}_1, \tilde{t}_2]$. $\tau(t)$ is not in sliding manifold during $[\tilde{t}_1, \tilde{t}_2)$. Hence, $\exists$ some integer l such that $l\gamma_0 < \tau(t) < (l+1)\gamma_0$. Otherwise, sliding mode happens in $\tau(t) = l\gamma_0$ or $\tau(t) = (l+1)\gamma_0$. Hence, $|\tilde{\tau}(t)| = |\tau(t) - \tau(\tilde{t}_1)| = O(\gamma_0)$, $\forall t \in [\tilde{t}_1, \tilde{t}_2)$. Since $0 < |\tanh(\sin(\pi\tau(t)/\gamma_0))| \leq 1$, $\forall t \in [\tilde{t}_1, \tilde{t}_2)$, one has $|v(t)| \geq |\phi \tanh^2(\sin(\pi\tau(t)/\gamma_0))|$. They guarantee $\alpha > 0$ so that $|\dot{\tau}(t)| \geq \tilde{k}(|\phi \tanh^2(\sin(\pi\tau/\gamma_0))| - |\omega|) \geq \tilde{\alpha}$ in which $\tilde{\alpha} = \tilde{k}\alpha > 0$. Thus, $(t - \tilde{t}_1) \leq |\tilde{\tau}|/\tilde{\alpha}$, $\forall t \in [\tilde{t}_1, \tilde{t}_2)$ and $(t - \tilde{t}_1)$ is of order $O(\gamma_0)$, $\forall t \in [\tilde{t}_1, \tilde{t}_2)$. Thus, one can assure that $y(t) - y(\tilde{t}_1)$ is of order $O(\gamma_0)$, $\forall t \in [\tilde{t}_1, \tilde{t}_2)$. By continuity, $|y(t) - y(\tilde{t}_1)| = O(\gamma_0)$, $\forall t \in [\tilde{t}_1, \tilde{t}_2]$.

For (II), let $t \in [\tilde{t}_1, \tilde{t}_3]$. Therefore, we have that $y(t) - y(\tilde{t}_1)$ is of order $O(\gamma_0)$, $\forall t \in [\tilde{t}_1, \tilde{t}_2]$. Next, consider $t \in [\tilde{t}_2, \tilde{t}_3]$. Since $\tau(t)$ is in sliding motion during $[\tilde{t}_2, \tilde{t}_3]$, one has $\dot{\tau}(t) = 0$. $y(t)$, $\forall t \in [\tilde{t}_2, \tilde{t}_3]$ is increasing with $\dot{y} = k_r - (\gamma_2/\gamma_1)_0 D_t^q \mathrm{sgn}(e) > 0$. Hence, $y(t)$ continues to approach y^* during $[\tilde{t}_2, \tilde{t}_3]$. So, $y(t) - y(\tilde{t}_1)$ is of order $O(\gamma_0)$, $\forall t \in [\tilde{t}_2, \tilde{t}_3]$. One has that the oscillation outside D_δ is of order $O(\gamma_0)$, $\forall t \in [\tilde{t}_1, \tilde{t}_3]$. The boundedness of y means that θ is UB, from the continuity of $y = F(\theta)$. Although $\lim_{t \to +\infty} |\tau(t)| = +\infty$ (actually, it is just an argument of an FO sign function in (5)), it is not bad for UB of other signals. One can derive that other closed-loop signals are UB.

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