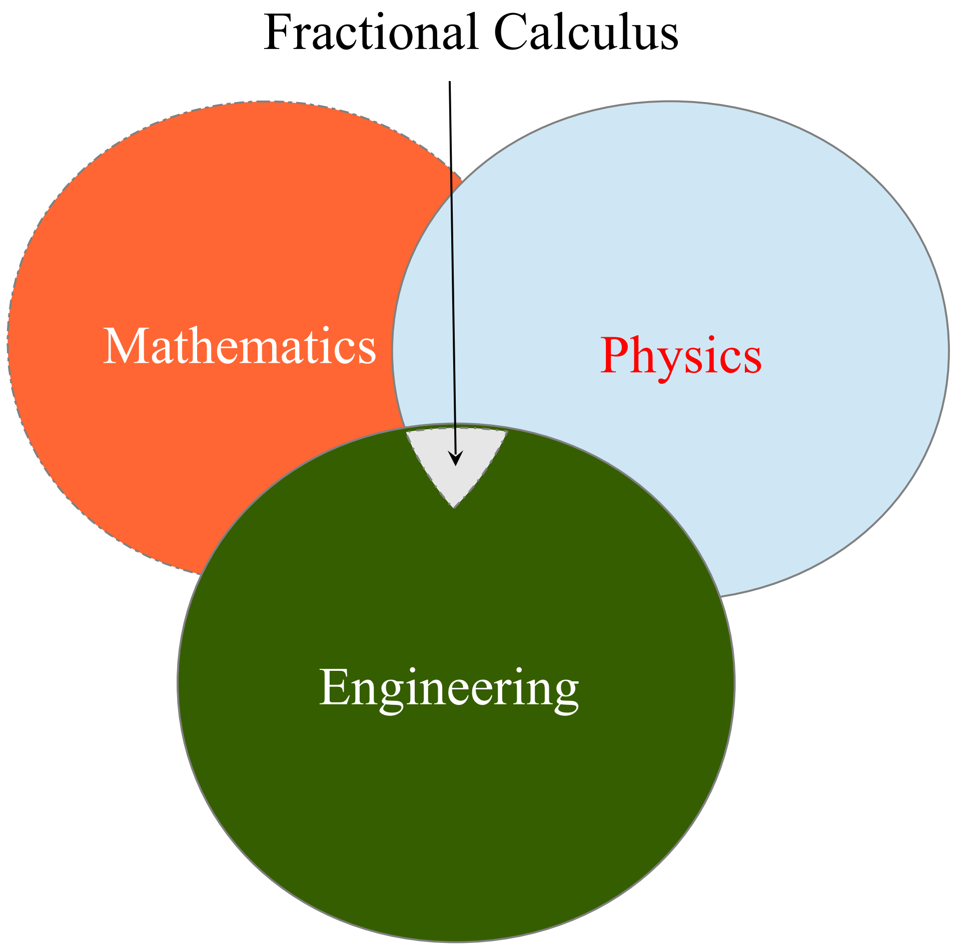
Random walk through Anomalous processes: some applications of Fractional calculus

Nickolay Korabel

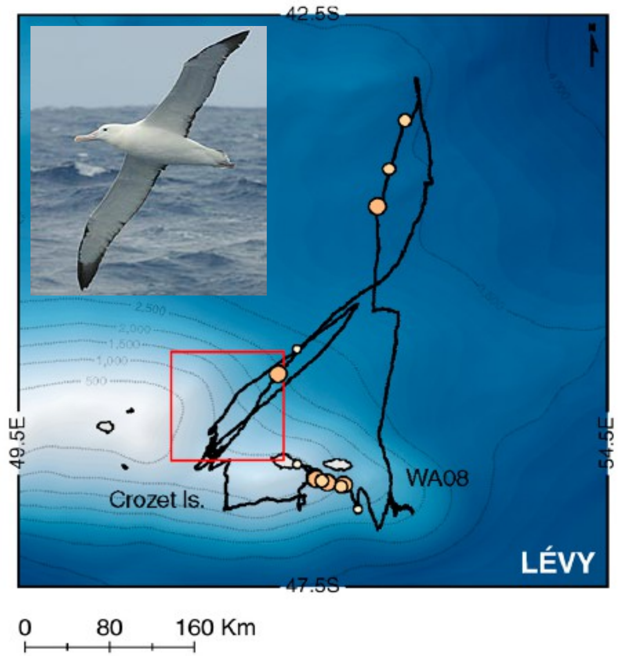
UNIVERSITY OF CALIFORNIA UCMERCED

Fractional Calculus Day @ UCMerced

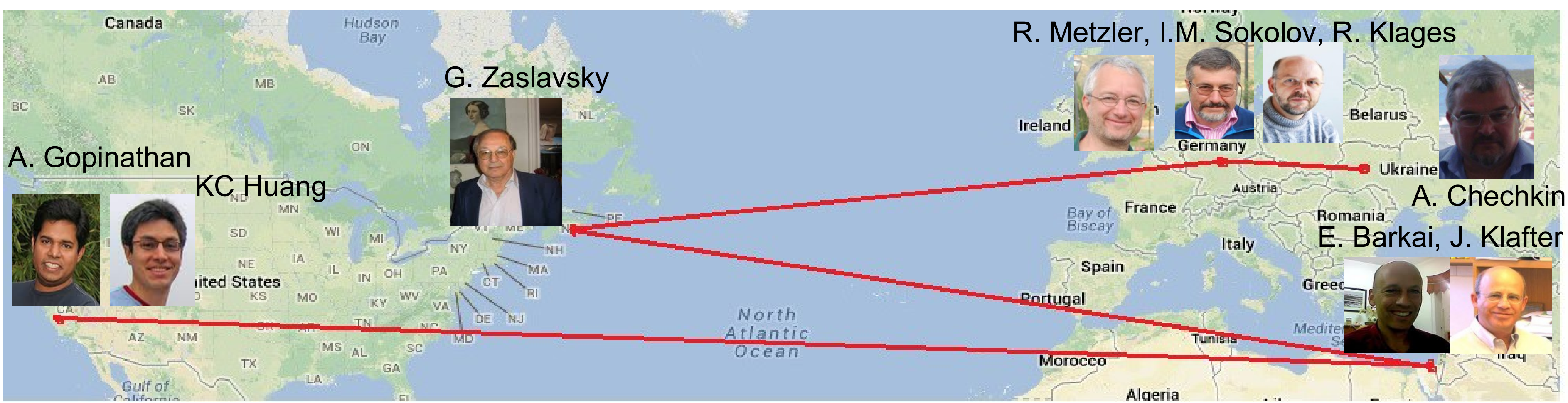
12 June 2013


Fractional Calculus
Mathematics
Physics
Engineering

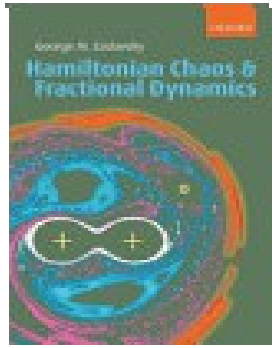
Lévy flights for foraging and knowledge

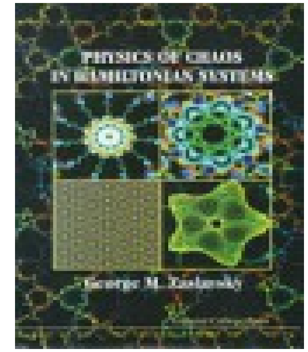


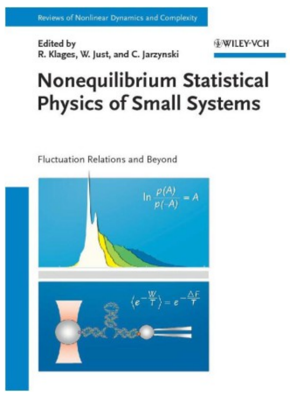
Humphries et al PNAS (2012)

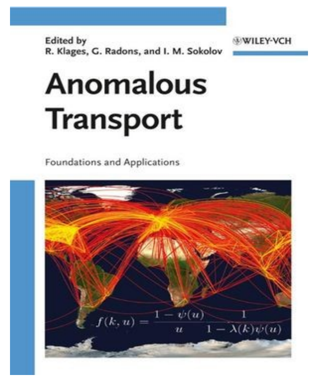


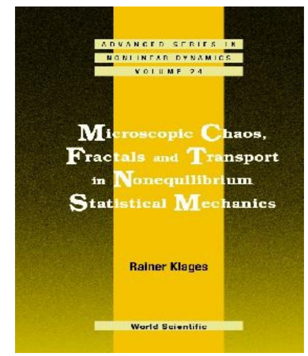
My great teachers

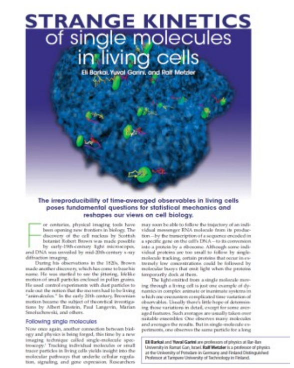




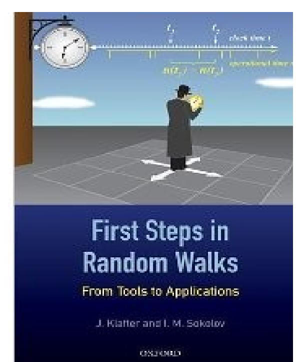




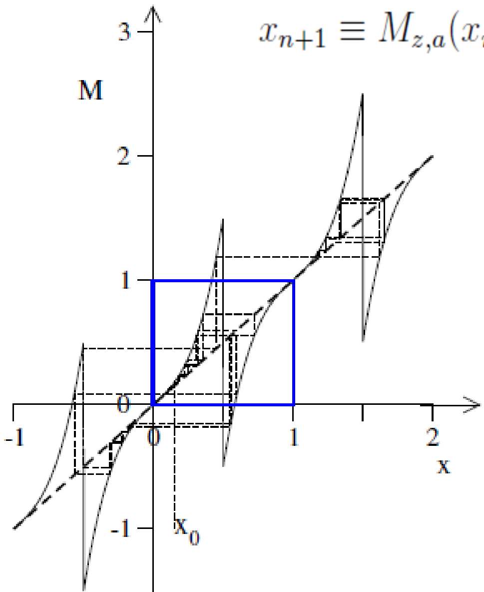



STRANGE KINETICS
of single molecules in living cells
Eli Barkai, Yuval Garini, and Ralf Metzler

The irreproducibility of time-averaged observables in living cells poses fundamental questions for statistical mechanics and reshapes our views on cell biology.

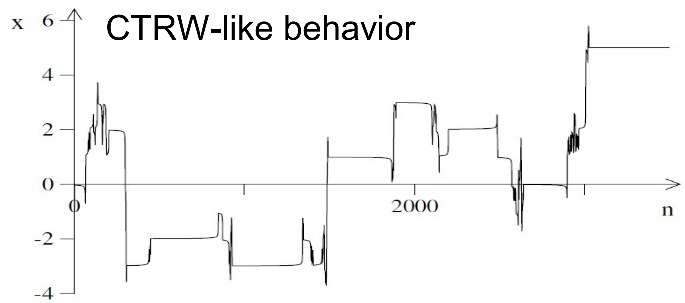


Sub-diffusive map



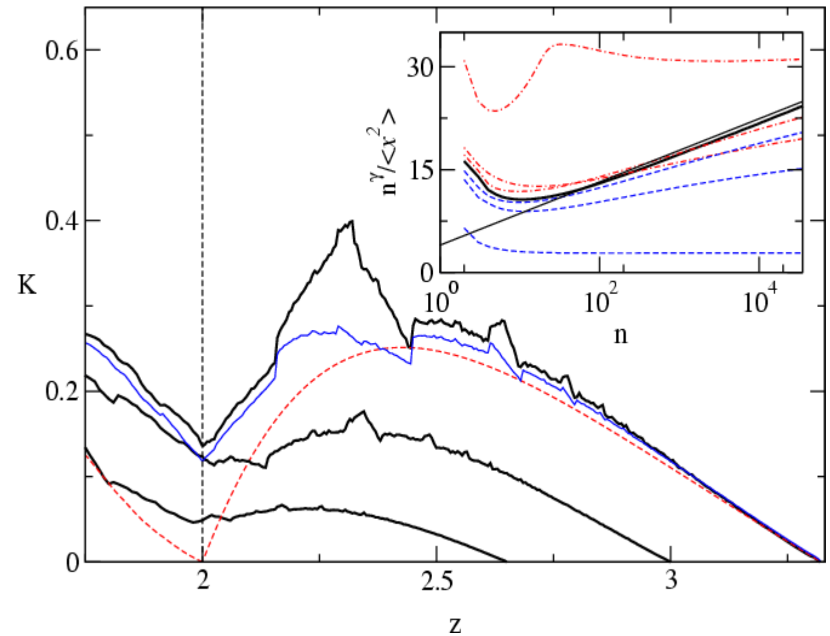
$$x_{n+1} \equiv M_{z,a}(x_n) = x_n + a x_n^z, \quad 0 < x_n < \frac{1}{2}$$

CTRW-like behavior



$$K = \lim_{n\to\infty} \frac{\langle x^2 \rangle}{n^\alpha}$$

$$\alpha = \begin{cases} 1, & z < 2 \\ \frac{1}{z-1}, & z \geq 2. \end{cases}$$



NK, R. Klages, Chechkin, Sokolov

Time fractional Fokker-Plank equation for sub-diffusion

Montroll-Weiss equation

$$\tilde{\hat{P}}(k,s) = \frac{1-\hat{\lambda}(k=0)\tilde{w}(s)}{s}\frac{1}{1-\hat{\lambda}(k)\tilde{w}(s)}.$$

jump length pdf

waiting time pdf

$$\tilde{\hat{P}}(k,s) = \frac{cb^{\gamma}s^{\gamma-1}}{\frac{p(kl)^2}{2} + c(bs)^{\gamma}}$$

See works of F. Mainardi and R. Gorenflo

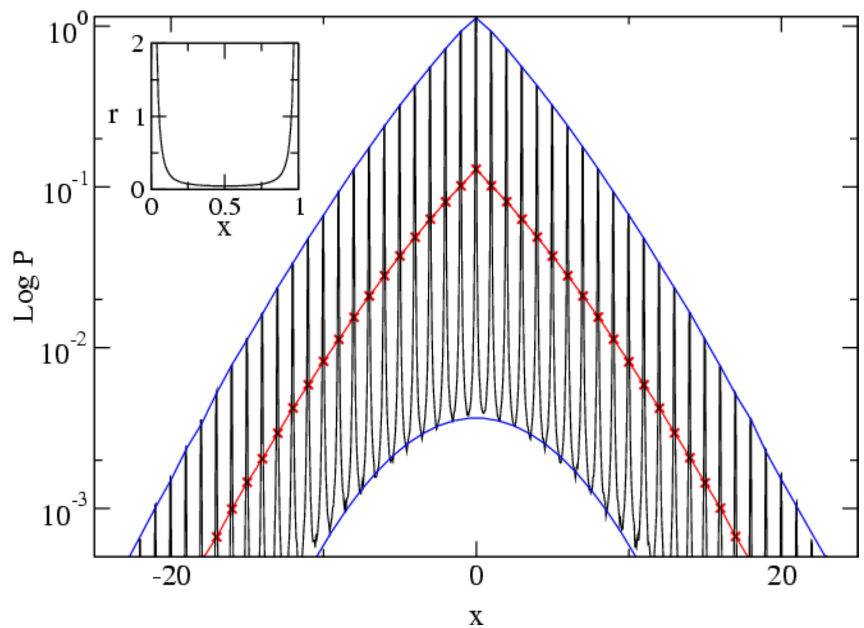
$$s^{\gamma}\tilde{\hat{P}} - s^{\gamma-1} = -\frac{pl^2}{2cb^{\gamma}}k^2\tilde{\hat{P}}$$

Caputo fractional derivative

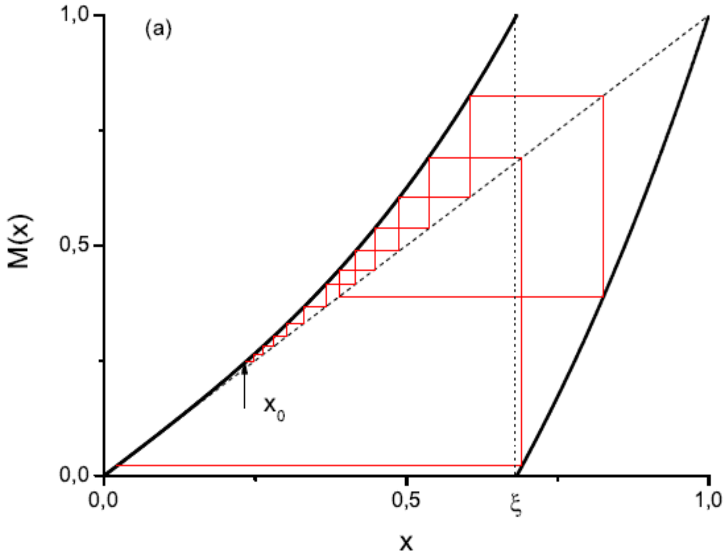
$$\frac{\partial^{\gamma}G}{\partial t^{\gamma}} \equiv \frac{1}{\Gamma(1-\gamma)}\int_0^t dt'(t-t')^{-\gamma}\frac{\partial G}{\partial t'}$$

$$\int_0^{\infty} dt\, e^{-st}\frac{\partial^{\gamma}G}{\partial t^{\gamma}} = s^{\gamma}\tilde{G}(s) - s^{\gamma-1}G(0)$$

$$\frac{\partial^{\gamma}P(x,t)}{\partial t^{\gamma}} = D\frac{\partial^2 P}{\partial x^2}$$



Infinite invariant density

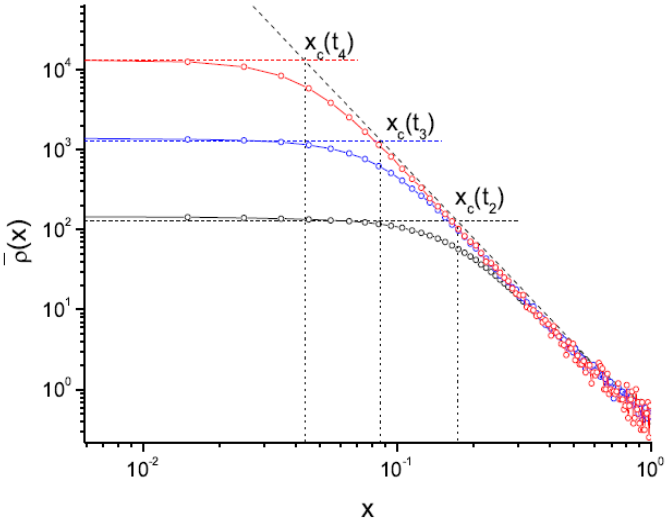


$$M(x_t) = x_t + a x_t^z \ (\mathrm{mod}.1), \quad z \geq 1, \ a > 0$$

$$\frac{\partial \rho_c(x,t)}{\partial t} = -\frac{\partial}{\partial x}(a x^z \rho_c(x,t)) + a\xi^z \rho_c(\xi,t)$$

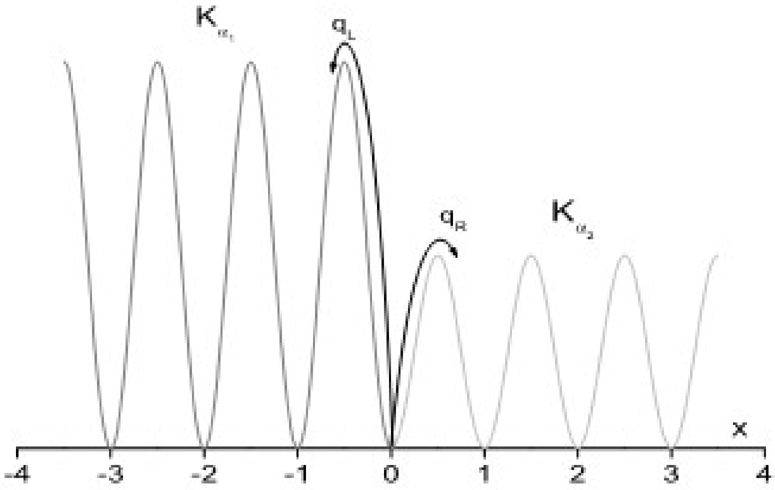
$$\rho_c(x,t) \sim \begin{cases} \frac{a^{\alpha-1} x^{-\frac{1}{\alpha}}}{\alpha^\alpha} \frac{\sin(\pi\alpha)}{\pi} t^{\alpha-1}, & x \gg x_c \\ \frac{\sin(\pi\alpha)}{\pi\alpha^{1+\alpha}} t^\alpha, & x \ll x_c. \end{cases}$$

$$\bar{\rho}_c(x) = t^{1-\alpha} \rho_c(x,t) = \begin{cases} \frac{a^{\alpha-1}}{\alpha^\alpha} \frac{\sin(\pi\alpha)}{\pi} x^{-\frac{1}{\alpha}}, & x \gg x_c \\ \frac{\sin(\pi\alpha)}{\pi\alpha^{1+\alpha}} t, & x \ll x_c, \end{cases}$$



NK, E. Barkai

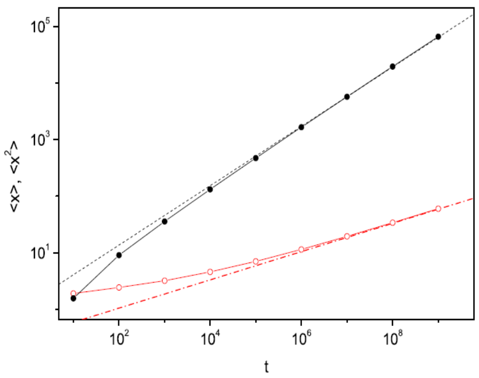
Paradoxes of sub-diffusion: anomalous infiltration

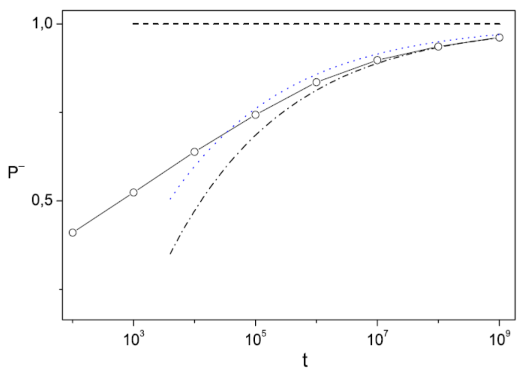


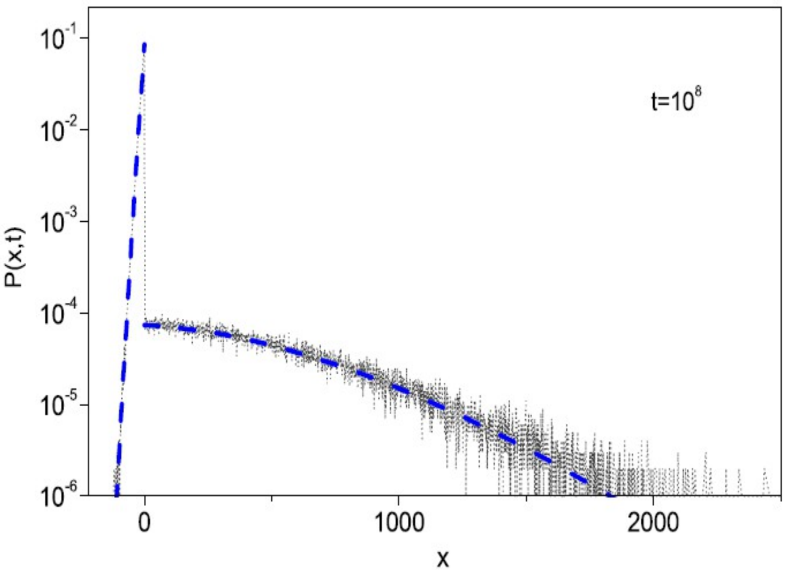
$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\alpha^-} K^- \frac{\partial^2}{\partial x^2} P(x,t), \qquad x<0,$$
$$\frac{\partial P(x,t)}{\partial t} = {}_0D_t^{1-\alpha^+} K^+ \frac{\partial^2}{\partial x^2} P(x,t), \qquad x>0,$$

Riemann–Liouville operator:

$${}_0D_t^{1-\alpha} P(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \mathrm{d}t' \frac{P(x,t')}{(t-t')^{1-\alpha}}$$







NK, E. Barkai

Fractional equations with long range interactions
Non-linear Schroedinger equations

Continuous non-linear Schroedinger equations: $$i\frac{d\psi}{dt} + \gamma|\psi|^2\psi + \frac{\partial^2\psi}{\partial x^2} = 0$$

Discrete lattice of coupled osccillators: $$i\frac{d\psi_n}{dt} + \gamma|\psi_n|^2\psi_n + \epsilon\left(\psi_{n+1} + \psi_{n-1} - 2\psi_n\right) = 0$$

Conserved quantities: $$H = -i\sum_{n=1}^{N}\left(\epsilon|\psi_{n+1} - \psi_n|^2 - \gamma|\psi_n|^4\right) \quad M = \sum_{n=1}^{N}|\psi_n|^2$$

Stationary solutions in the form: $\psi_n(t) = \phi_n \exp(i\omega t)$

$$-\omega\phi_n + \gamma|\phi_n|^2\phi_n + \epsilon\left(\phi_{n+1} + \phi_{n-1} - 2\phi_n\right) = 0$$

For large N DNLS is not integrable and chaotic solutions are possible.

Osccillators with all-to-all long range interactions: $$H = T + U = \frac{1}{2}\sum_{\substack{n,m=1\\ n\neq m}}^{N} J_{n-m}|\psi_m - \psi_n|^2 - \frac{1}{2}\sum_{n=1}^{N}|\psi_n|^4$$

$$J_{n-m} = J/|n-m|^{1+\alpha}$$

Equations of motion: $$\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t} + \gamma|\psi_n|^2\psi_n + \sum_{\substack{m=1\\ m\neq n}}^{N} J_{n-m}(\psi_n - \psi_m) = 0$$

NK, G. Zaslavsky, V. Tarasov, N. Laskin

Fractional equations with long range interactions: Non-linear Schroedinger equation

Equations of motion:

$$\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t} + \gamma|\psi_n|^2\psi_n + \sum_{\substack{m=1\\ m\neq n}}^{N} J_{n-m}(\psi_n - \psi_m) = 0$$

$$J_{n-m} = J/|n-m|^{1+\alpha}$$

Transition to continuous equation: $\hat{\psi}_n(k,t) = \sum_{n=-\infty}^{\infty} \psi_n(t)\exp(-\mathrm{i}kx_n) \equiv \mathscr{F}_\Delta\{\psi_n(t)\}$

$$\mathrm{i}\frac{\partial\hat{\psi}(k,t)}{\partial t} + \gamma\,\mathscr{F}_\Delta\{|\psi_n|^2\psi_n\} + J\,(\hat{J}_\alpha(k) - \hat{J}_\alpha(0))\hat{\psi}(k,t) = 0$$

$$\hat{J}_\alpha(k) = \sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}kn\Delta x}}{|n|^{1+\alpha}} = \sum_{n=1}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}kn\Delta x} + \mathrm{e}^{\mathrm{i}kn\Delta x}}{n^{1+\alpha}} = Li_{1+\alpha}(\mathrm{e}^{\mathrm{i}k\Delta x}) + Li_{1+\alpha}(\mathrm{e}^{-\mathrm{i}k\Delta x})$$

Polylogarithmic function

$$\hat{J}_\alpha(k) = a_\alpha|\Delta x|^\alpha|k|^\alpha + 2\sum_{n=0}^{\infty}\frac{\zeta(1+\alpha-2n)}{(2n)!}(\Delta x)^{2n}(-k^2)^n, \quad |k|<1, \quad \alpha\neq 0,1,2,3,\ldots,$$

$$\mathrm{i}\frac{\partial\hat{\psi}(k,t)}{\partial t} + \bar{J}\hat{\mathscr{T}}_{\alpha,\Delta}(k)\,\hat{\psi}(k,t) + \gamma\mathscr{F}_\Delta\{|\psi_n|^2\psi_n\} = 0, \qquad \hat{\mathscr{T}}_{\alpha,\Delta}(k) = \begin{cases} a_\alpha|k|^\alpha - |\Delta x|^{2-\alpha}\zeta(\alpha-1)k^2, & 0<\alpha<2 \quad (\alpha\neq 1),\\ |\Delta x|^{\alpha-2}a_\alpha|k|^\alpha - \zeta(\alpha-1)k^2, & 2<\alpha<4 \quad (\alpha\neq 3).\end{cases}$$

$$\mathrm{i}\frac{\partial}{\partial t}\psi(x,t) + \bar{J}\mathscr{T}_\alpha(x)\psi(x,t) + \gamma|\psi(x,t)|^2\psi(x,t) = 0 \quad \alpha\neq 0,1,2,\ldots,$$

$$\mathscr{T}_\alpha(x) = \mathscr{F}^{-1}\{\hat{\mathscr{T}}_\alpha(k)\} = \begin{cases} -a_\alpha\frac{\partial^\alpha}{\partial|x|^\alpha}, & 0<\alpha<2, \quad (\alpha\neq 1),\\ \zeta(\alpha-1)\frac{\partial^2}{\partial x^2}, & \alpha>2, \quad (\alpha\neq 2,3,4,\ldots);\end{cases}$$

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$$|k|^\alpha \longleftrightarrow -\frac{\partial^\alpha}{\partial|x|^\alpha}, \quad k^2 \longleftrightarrow -\frac{\partial^2}{\partial x^2}$$

Fractional equations with long range interactions: Non-linear Schroedinger equation

$$i\frac{\partial}{\partial t}\psi(x,t) + \bar{J}\mathcal{T}_\alpha(x)\psi(x,t) + \gamma|\psi(x,t)|^2\psi(x,t) = 0 \quad \alpha \neq 0, 1, 2, \dots,$$

$$\mathcal{T}_\alpha(x) = \mathcal{F}^{-1}\{\hat{\mathcal{T}}_\alpha(k)\} = \begin{cases} -a_\alpha \frac{\partial^\alpha}{\partial|x|^\alpha}, & 0<\alpha<2, \quad (\alpha\neq 1), \\ \zeta(\alpha-1)\frac{\partial^2}{\partial x^2}, & \alpha>2, \quad (\alpha\neq 2,3,4,\dots); \end{cases}$$

To visualize numerical results we use:

Surface - $|\psi(x,t)|^2$ Power spectum: $S_j \equiv S(w_j) = |\hat{\psi}_n(w_j)|^2$

Phase portrait of the central osccillator: $A(t) = |\psi(0,t)|^2, \quad A_t = \mathrm{d}A/\mathrm{d}t$

Amplitude of central osccillator: $z = \psi(0,t)$

Initial conditions: $\psi(x,0) = a + b\cos\left(\frac{2\pi}{L}x\right)$ $\psi(x,0) = a\left[1 + b\left\{e^{ic}\cos\left(\frac{2\pi}{L}x\right) + e^{id}\sin\left(\frac{2\pi}{L}x\right)\right\}\right]$

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Fractional equations with long range interactions
Non-linear Schroedinger equation

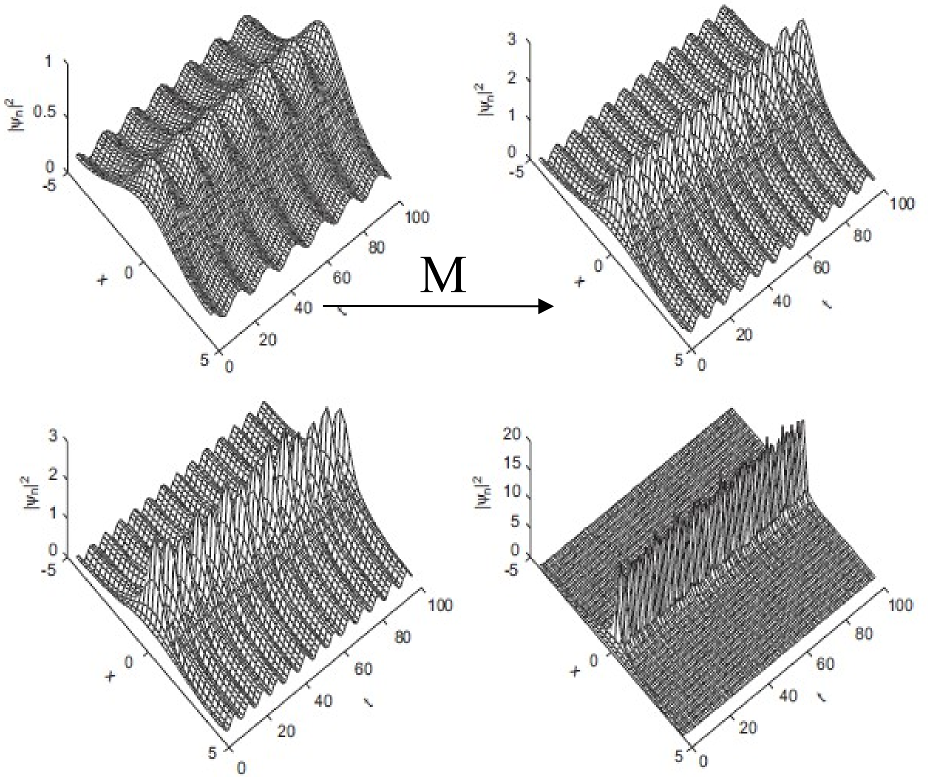
$$\boxed{\mathrm{i}\frac{\partial}{\partial t}\psi(x,t) + \bar{J}\mathscr{T}_{\alpha}(x)\psi(x,t) + \gamma|\psi(x,t)|^2\psi(x,t) = 0 \quad \alpha \neq 0, 1, 2, \ldots,}$$

$$\mathscr{T}_{\alpha}(x) = \mathscr{F}^{-1}\{\hat{\mathscr{T}}_{\alpha}(k)\} = \begin{cases} -a_{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}, & 0 < \alpha < 2, \quad (\alpha \neq 1), \\ \zeta(\alpha-1)\frac{\partial^2}{\partial x^2}, & \alpha > 2, \quad (\alpha \neq 2, 3, 4, \ldots); \end{cases}$$

To visualize numerical results we use:

Surface - $|\psi(x,t)|^2$

$\alpha = 1.11$



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Fractional equations with long range interactions
Non-linear Schroedinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi(x,t)+\bar{J}\mathscr{T}_{\alpha}(x)\psi(x,t)+\gamma|\psi(x,t)|^{2}\psi(x,t)=0 \quad \alpha\neq 0,1,2,\ldots,$$

$$\mathscr{T}_{\alpha}(x)=\mathscr{F}^{-1}\{\hat{\mathscr{T}}_{\alpha}(k)\}=\begin{cases} -a_{\alpha}\frac{\partial^{\alpha}}{\partial|x|^{\alpha}}, & 0<\alpha<2, \quad (\alpha\neq 1), \\ \zeta(\alpha-1)\frac{\partial^{2}}{\partial x^{2}}, & \alpha>2, \quad (\alpha\neq 2,3,4,\ldots); \end{cases}$$

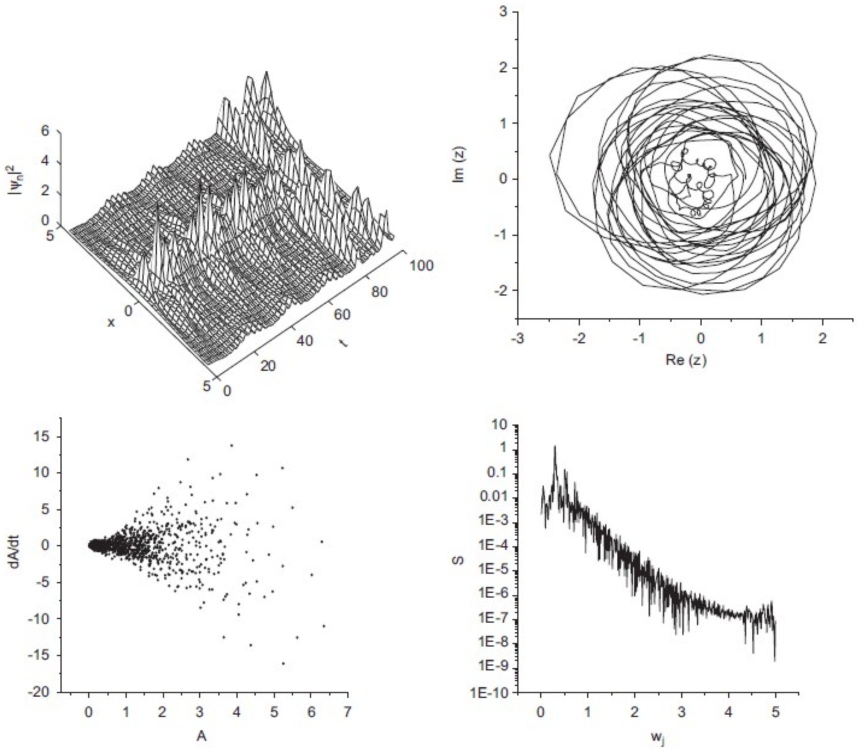

Fig. 7. Solution of the standard DNLS equation with $M = 12.5$ and $J/J_0 = 0.7$. The initial condition given by Eq. (34).

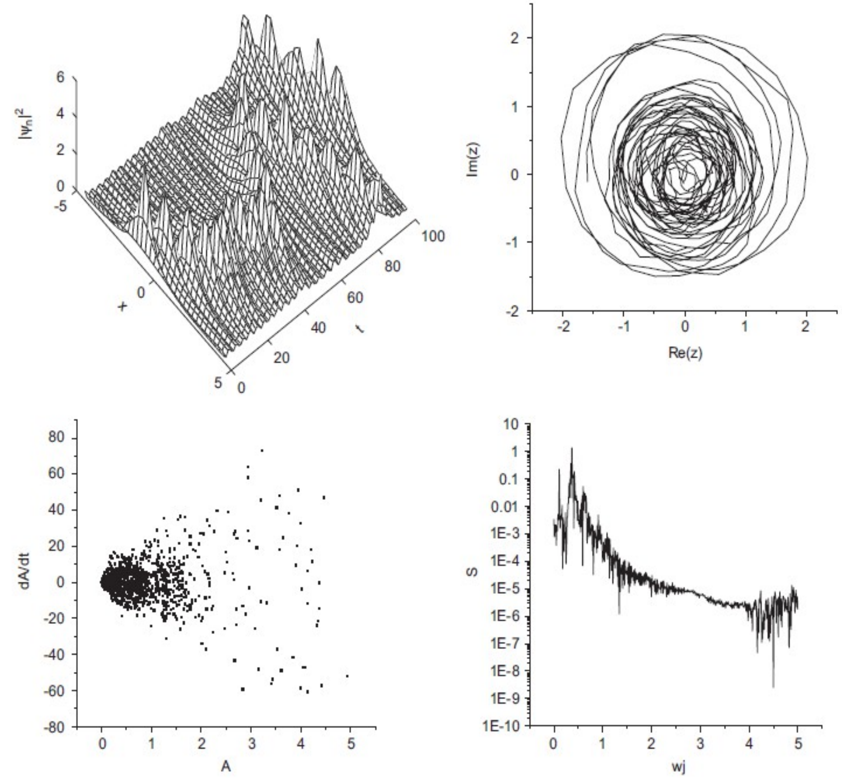

Fig. 8. Time evolution of the system of coupled oscillators with LRI and asymmetric initial conditions. The values of parameters are $\alpha = 1.11$, $M = 22.22$, $J/J_0 = 1$. The initial condition is given by Eq. (35).

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Fractional equations with long range interactions
Sine-Gordon Equation

$$H = \sum_{n=-\infty}^{+\infty} \left[\frac{M}{2}\dot{u}_n^2 + \frac{J_0}{2} \sum_{\substack{m=-\infty \\ n\neq m}}^{+\infty} \frac{1}{|n-m|^{1+\alpha}} u_n u_m + \frac{J_1}{2} u_n^2 + J_2 \left(1 - \cos\left(\frac{2\pi u_n}{a}\right)\right)\right]$$

$$\frac{\partial^2 u_n}{\partial t^2} + J_0 \sum_{\substack{m=-\infty \\ n\neq m}}^{+\infty} \frac{1}{|n-m|^{1+\alpha}} u_m + J_1 u_n + J_2 \sin(u_n) = 0$$

$$\frac{\partial^2}{\partial t^2} u(x,t) + \bar{J}_0 \mathscr{T}_\alpha(x) u(x,t) + J_1 u(x,t) + J_2 \sin(u(x,t)) = 0 \quad \alpha \neq 0, 1, 2, \ldots,$$

$$\mathscr{T}_\alpha(x) = \mathscr{F}^{-1}\{\widehat{\mathscr{T}}_\alpha(k)\} = \begin{cases} -a_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha}, & 0 < \alpha < 2 \ (\alpha \neq 1), \\ \zeta(\alpha-1) \frac{\partial^2}{\partial |x|^2}, & 2 < \alpha < 4 \ (\alpha \neq 3); \end{cases}$$

$$\widehat{\mathscr{T}}_\alpha(k) = \begin{cases} a_\alpha |k|^\alpha, & 0 < \alpha < 2 \ (\alpha \neq 1), \\ -\zeta(\alpha-1) k^2, & 2 < \alpha < 4 \ (\alpha \neq 3). \end{cases}$$

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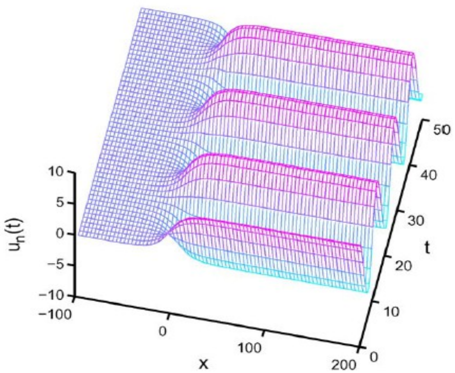
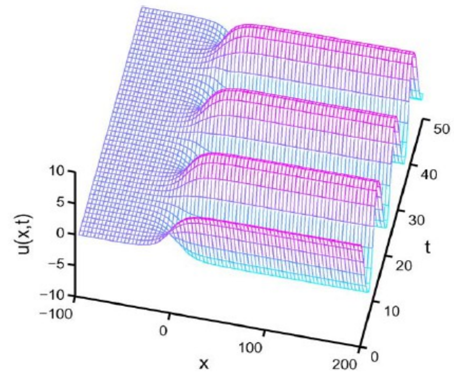
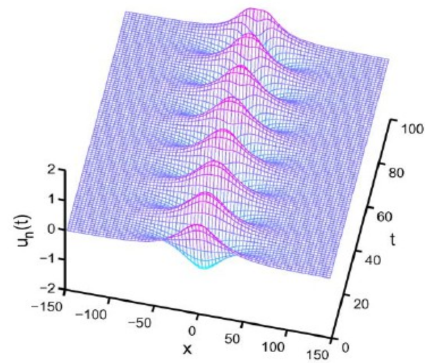
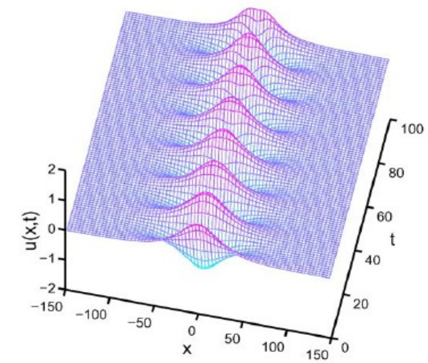
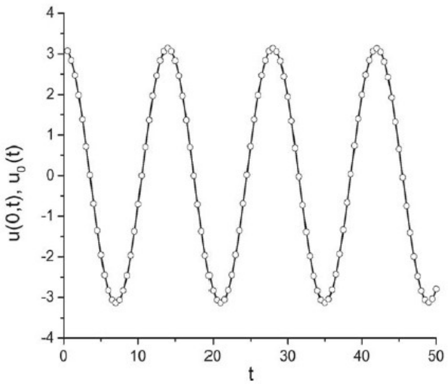
Fractional equations with long range interactions
Sine-Gordon Equation

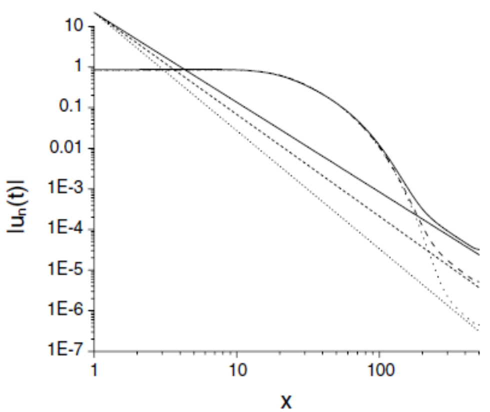


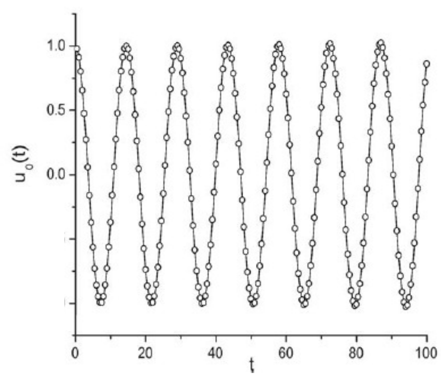












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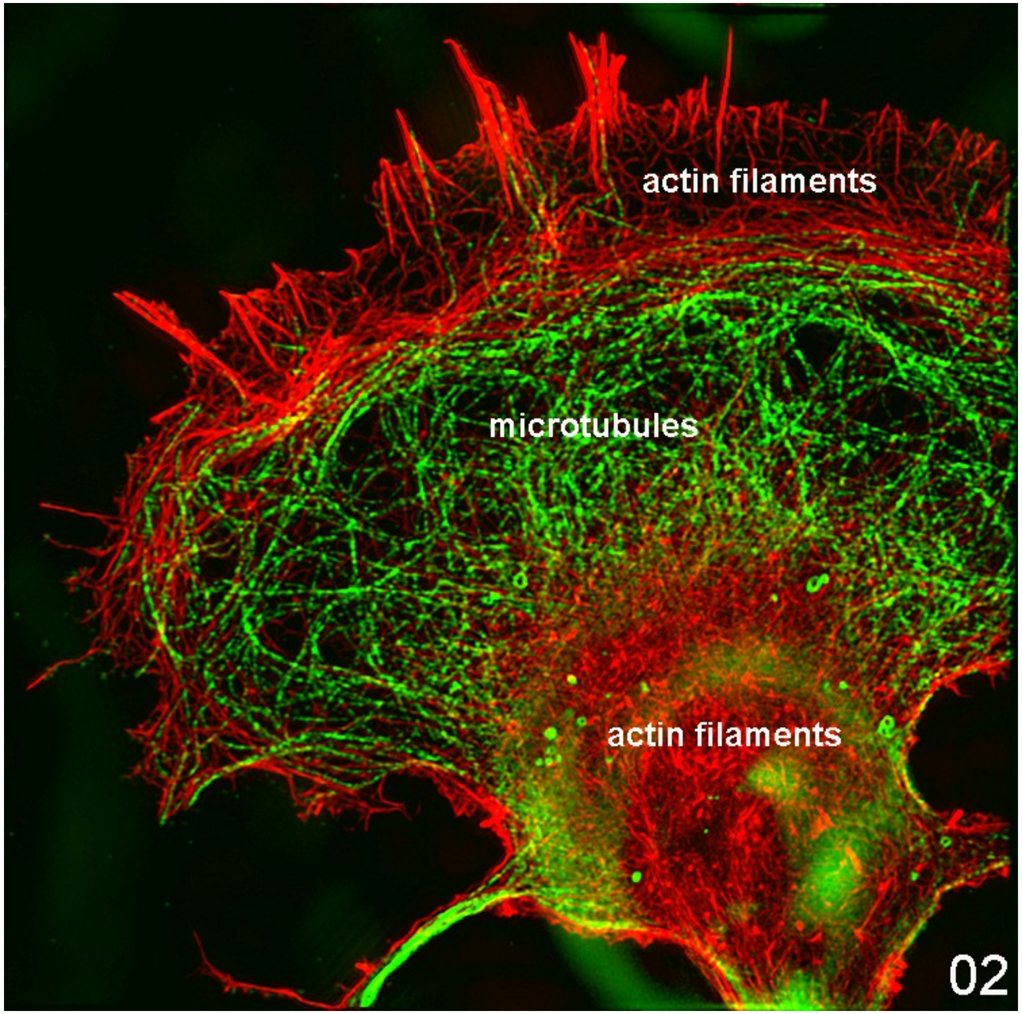
Modeling optimality in cytoskeleton transport

Ajay Gopinathan

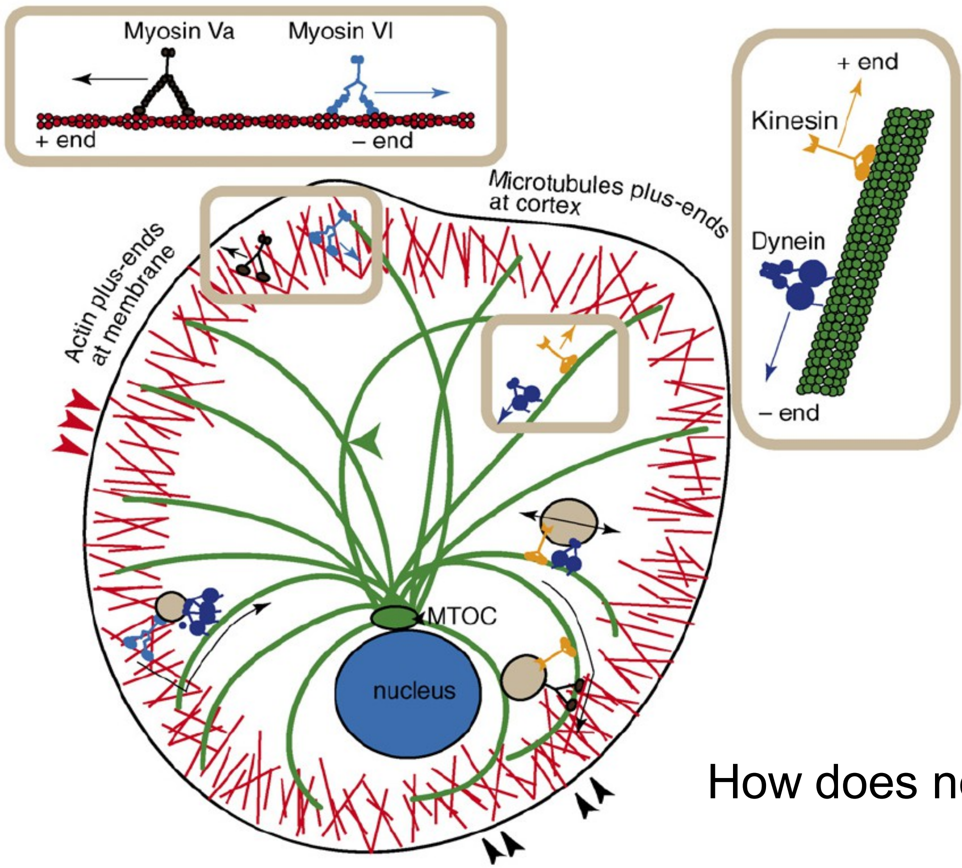
Kerwyn Casey Huang

Cytoskeleton network is required for structure, organization, and transport



Lammelipodium in a neuron

A complex cellular transportation system



Microtubules are like freeways and actin filaments are like local surface streets.

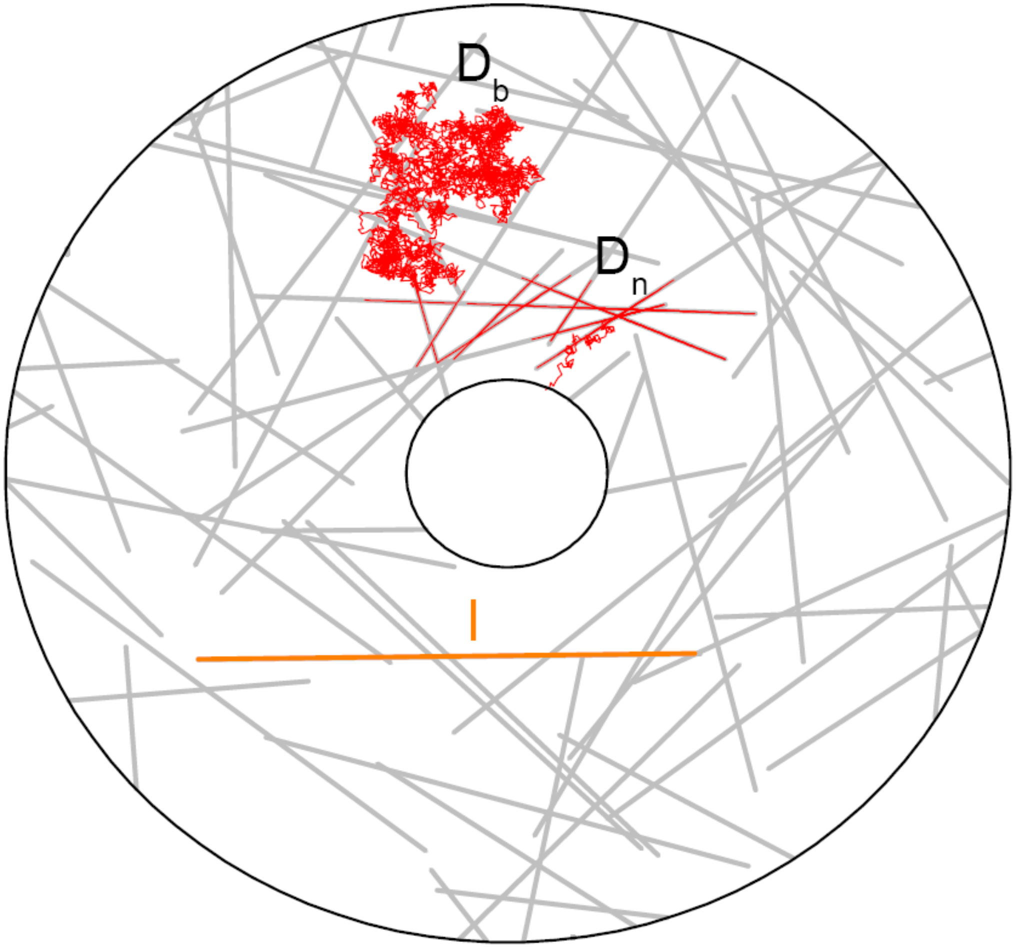
Organelles can move on both types of filaments.

Different types of motors work together.

How does network architecture influence transport?

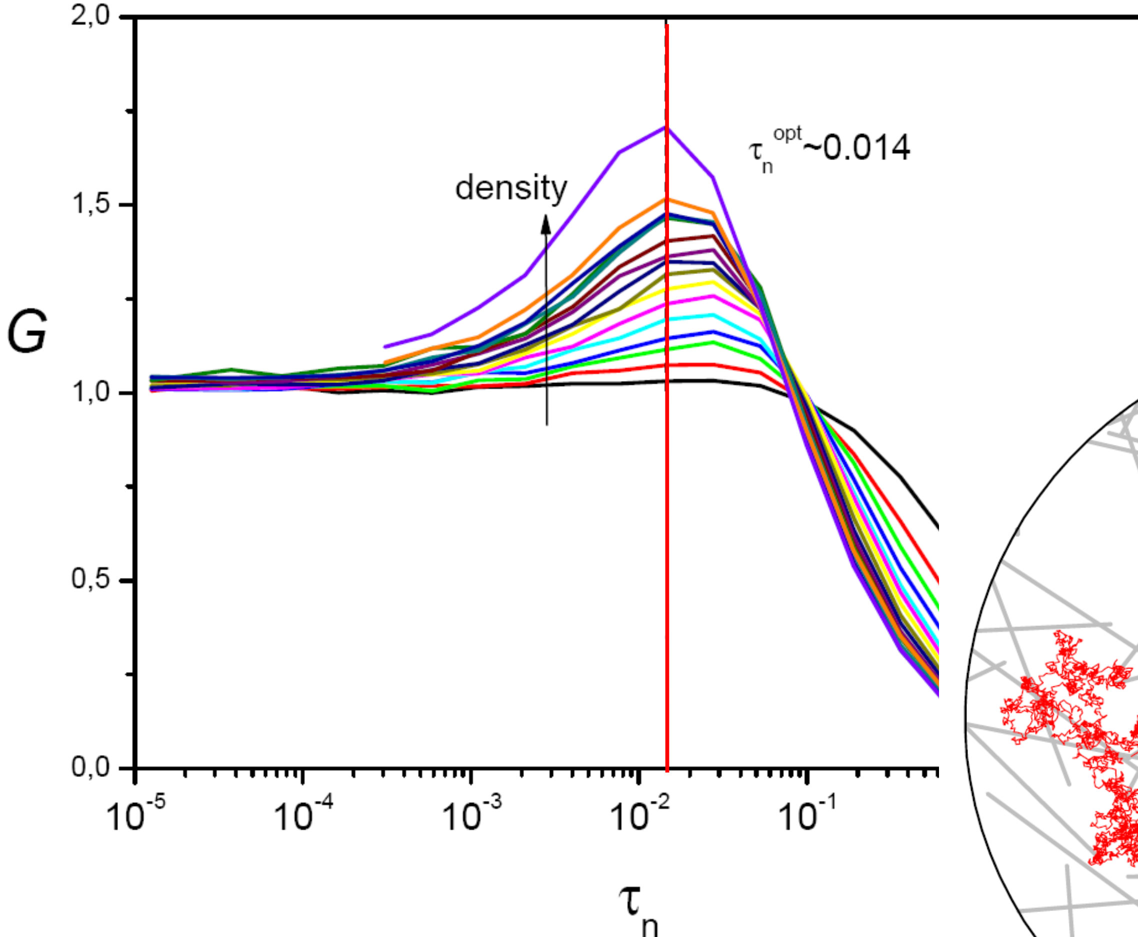
Are there optimal transport regimes in terms of network residence time, network density, filament length and orientation?

Continuum model of a cytoskeletal network



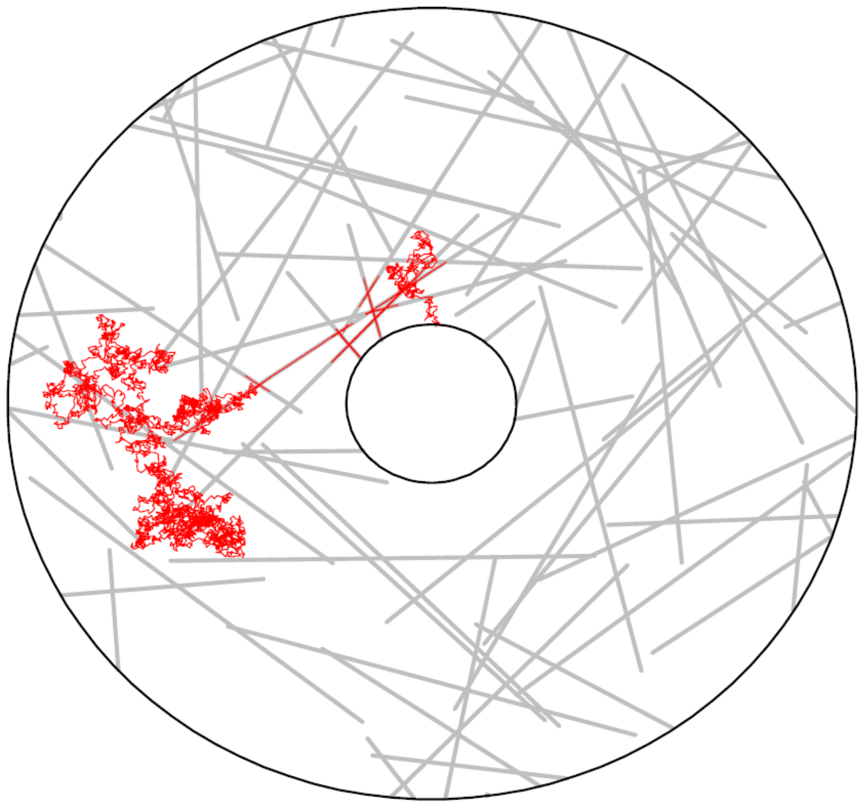
- Binding rate K_{on} (probability per unit time)
- Network residence time τ_n

There is an optimal network residence time



Diffusion constants in the bulk and on the network:

$$D_b = 0.1,\ D_n = 1$$



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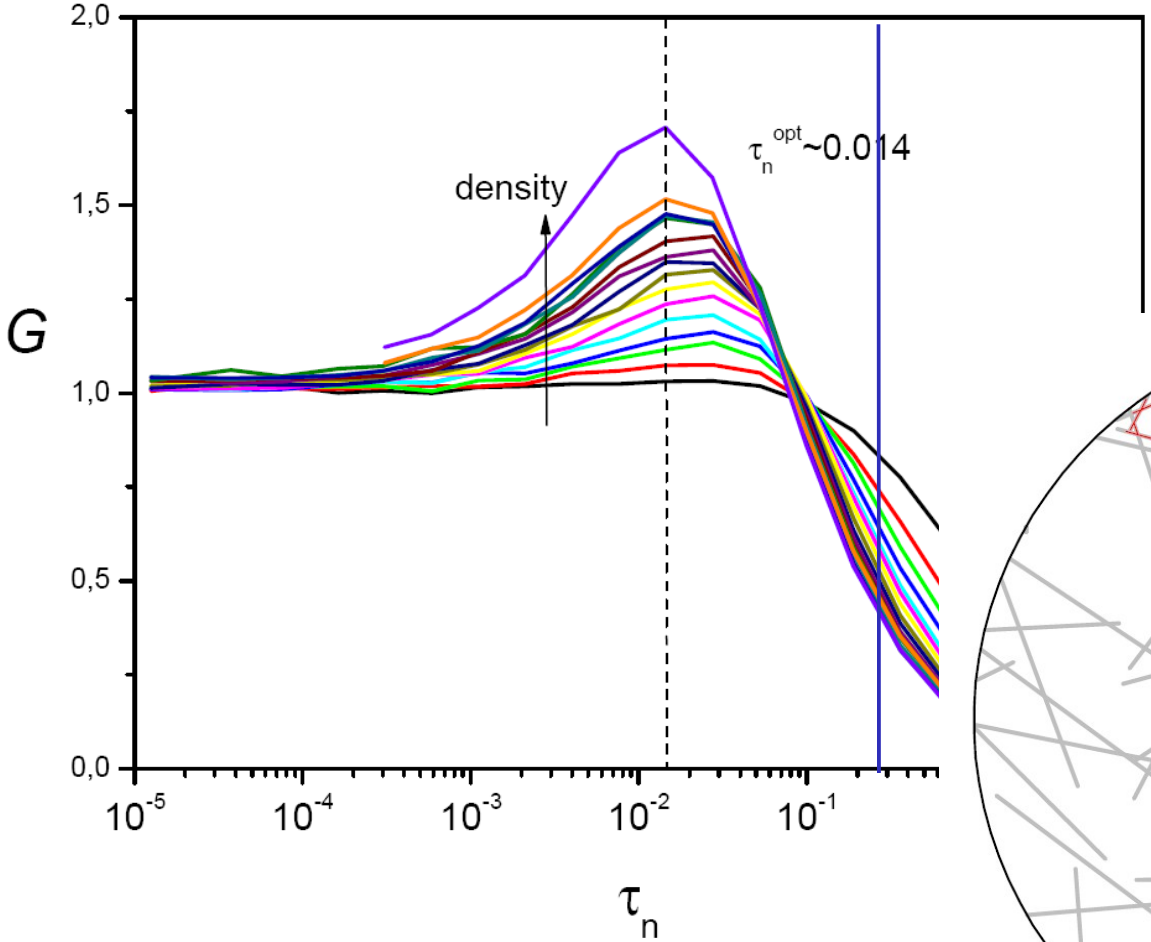
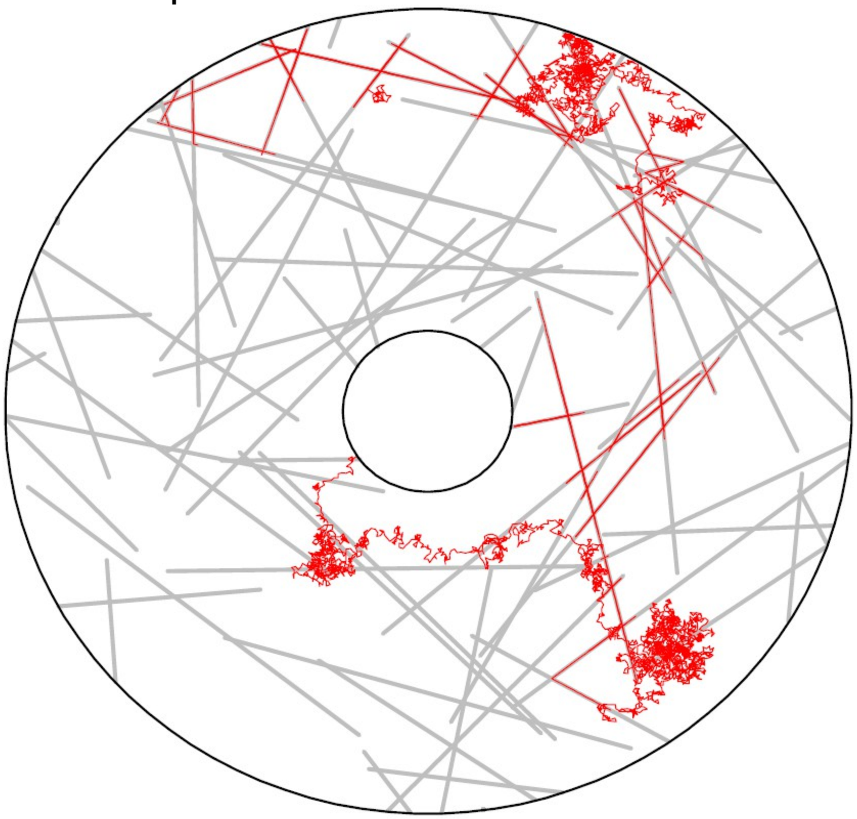
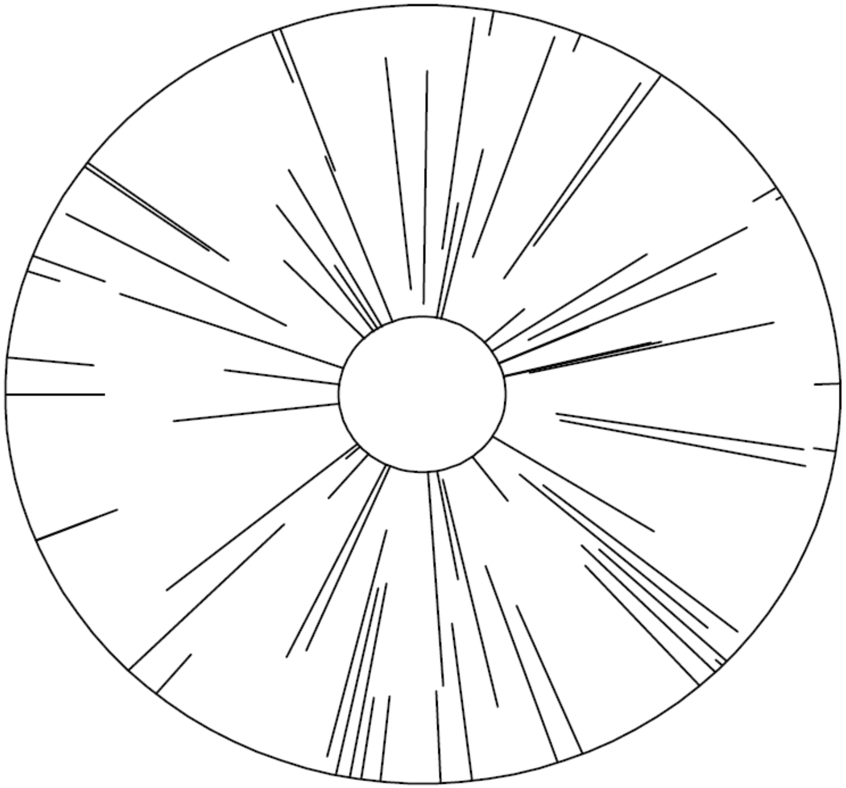


Diffusion constants in the bulk and on the network:

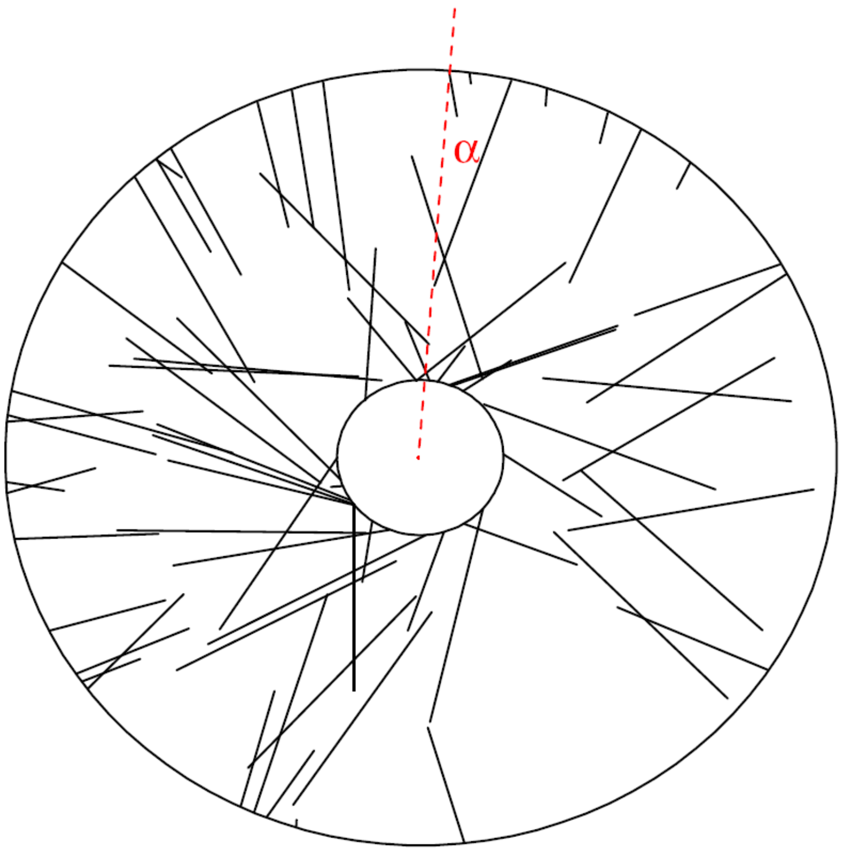
$$D_b = 0.1,\ D_n = 1$$



How network topology influences transport?

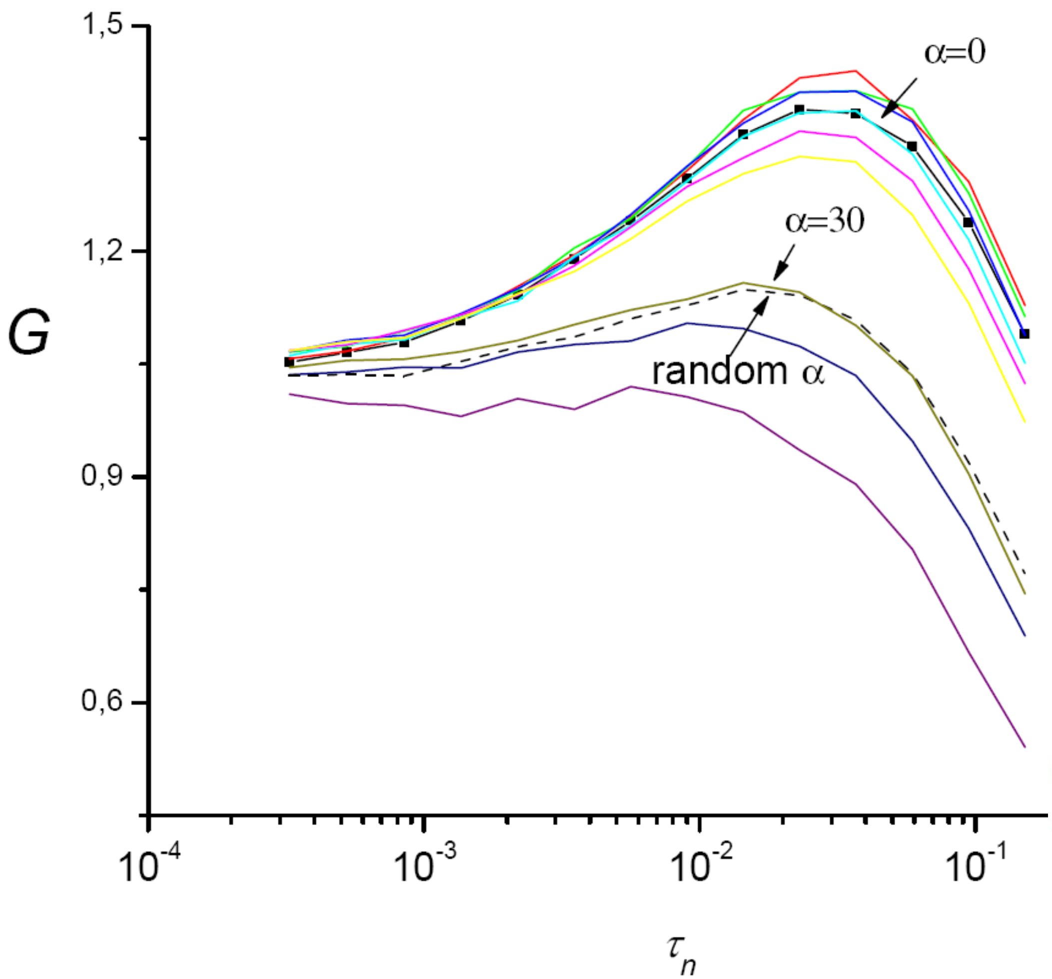


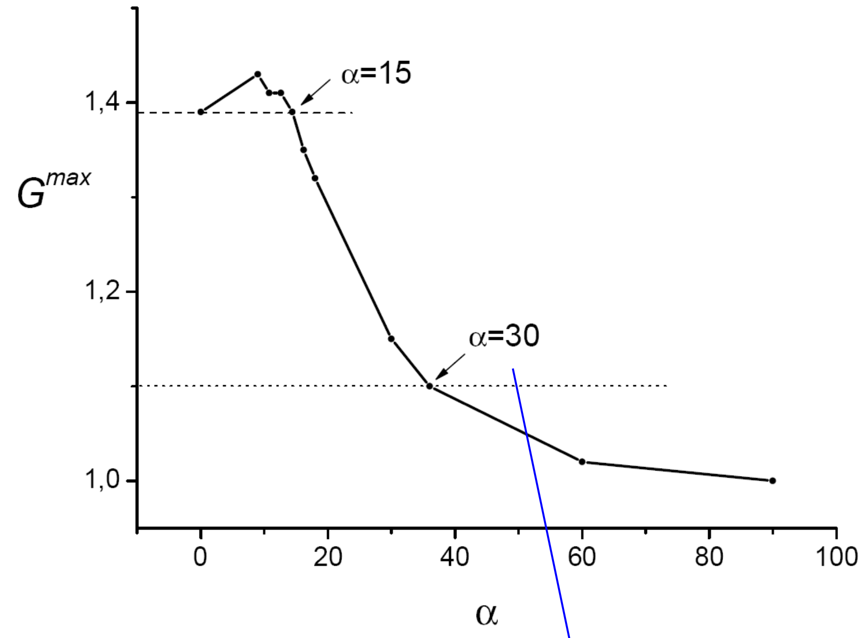
$\alpha = 0^\circ$

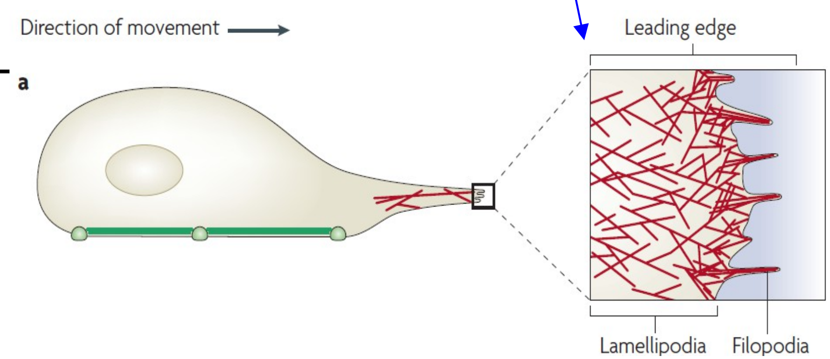


$\alpha = \pm 15^\circ$

Filament orientation affects transport







Call for Papers

Symposium on

Nonlinear Fractional Dynamics and Systems with Memory

To be held 5th International Conference on Nonlinear Science and Complexity

August 4-9, 2014 • Xi'an, P. R. of China

The symposium is to cover a broad scope of fractional nonlinear dynamics and dynamics of systems with memory in general (deterministic and stochastic). The fundamental theory and application in science and engineering are welcome. Manuscripts are solicited in the following topics but not restricted to:

- **General properties of fractional dynamical systems (solutions, attractors, stability, etc.) and systems with slow decay of correlations.**
- **Nonlinear fractional dynamics in physics (Levy flights and diffusion in Hamiltonian systems, materials with memory, dielectrics, etc.)**
- **Systems with memory in biology, psychology and neuroscience (brain, adaptation, human memory, neural networks)**
- **Fractional dynamics and systems with memory in social sciences (finance, economics, sociology).**
- **Circuit elements with memory.**
- **Nonlinear fractional control.**

The Conference website is http://nsc2014.xjtu.edu.cn (Under construction, it will be open in April 2013). For your convenience, we are attaching the first Call for Papers. The authors are encouraged to present a paper for publication in the edited books or conference Proceedings. The high quality papers will be selected for publication in *Journal of Applied Nonlinear Dynamics*. We look forward to hearing from you as soon as possible.

Paper Planning Schedule

Full paper submission:	**December, 1, 2013**
Notification deadline:	**March, 1, 2014**
Final paper submission:	**June, 1, 2014**

Email submission as a pdf file attachment is acceptable. Please transmit papers to the following organizers or submit it through the conference website: http://nsc2014.xjtu.edu.cn (Under construction, it will be open in April 2013)

Symposium Organizer:

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