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Fractional relaxation and diffusion equations of distributed order

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Essentials of Fractional Calculus in $\mathbf{R}^+$

For a sufficiently well-behaved function $f(t)$ ($t \in \mathbf{R}^+$) we may define the **fractional derivative** of order μ ($m-1 < \mu \le m\,,\ m \in \mathbf{N}$), see *e.g.* Gorenflo and Mainardi (1997), Podlubny (1999), in two different senses, that we refer here as to **Riemann-Liouville** (R-L) derivative and **Caputo** (C) derivative, respectively. Both derivatives are related to the so-called **Riemann-Liouville fractional integral** of order $\mu > 0$ defined as

$$ {}_tJ^{\mu}\, f(t) := \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1}\, f(\tau)\, d\tau\,, \quad \mu > 0\,, \qquad (A.1)$$

$$\Gamma(\mu) := \int_0^{\infty} \mathrm{e}^{-u} u^{\mu-1}\, du\,, \quad \Gamma(n+1) = n! \quad \textbf{Gamma function}.$$

By convention ${}_tJ^0 = I$ (Identity operator). We can prove

$$ {}_tJ^{\mu}\, {}_tJ^{\nu} = {}_tJ^{\nu}\, {}_tJ^{\mu} = {}_tJ^{\mu+\nu}\,,\ \mu, \nu \ge 0\,, \quad \textbf{semigroup property} \quad (A.2)$$

$$ {}_tJ^{\mu}\, t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\mu)}\, t^{\gamma+\mu}\,, \quad \mu \ge 0\,, \quad \gamma > -1\,, \quad t > 0\,. \qquad (A.3)$$

The fractional derivative of order $\mu > 0$ in the **Riemann-Liouville** sense is defined as the operator ${}_tD^\mu$ which is the **left inverse** of the Riemann-Liouville integral of order μ (in analogy with the ordinary derivative),

$$ {}_tD^\mu \, {}_tJ^\mu = I \,, \quad \mu > 0 \,. \qquad (A.4) $$

If m denotes the positive integer such that $m-1 < \mu \le m$, we recognize from Eqs. (A.2) and (A.4)

$$ {}_tD^\mu \, f(t) := {}_tD^m \, {}_tJ^{m-\mu} \, f(t) \,, \qquad (A.5) $$

hence

$$ {}_tD^\mu f(t) = \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\mu)}\displaystyle\int_0^t \dfrac{f(\tau)\,d\tau}{(t-\tau)^{\mu+1-m}}\right], & m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t), & \mu = m. \end{cases} \qquad (A.5') $$

For completion ${}_tD^0 = I$. The semigroup property is no longer valid but

$$ {}_tD^\mu \, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\mu)} \, t^{\gamma-\mu} \,, \quad \mu \ge 0 \,, \quad \gamma > -1 \,, \quad t > 0 \,. \qquad (A.6) $$

The property ${}_tD^\mu = {}_tJ^{-\mu}$ is not generally valid!

On the other hand, the fractional derivative of order $\mu \in (m-1, m]$ ($m \in \mathbf{N}$) in the **Caputo** sense is defined as the operator ${}_tD_*^{\mu}$ such that

$$ {}_tD_*^{\mu}\, f(t) := {}_tJ^{m-\mu}\, {}_tD^{m}\, f(t)\,, \qquad (A.7) $$

hence

$$ {}_tD_*^{\mu}\, f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\mu)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)\, d\tau}{(t-\tau)^{\mu+1-m}}\,, & m-1 < \mu < m, \\ \dfrac{d^m}{dt^m} f(t)\,, & \mu = m. \end{cases} \qquad (A.7') $$

Thus, when the order is not integer the two fractional derivatives differ in that the derivative of order m does not generally commute with the fractional integral.

We point out that the **Caputo fractional derivative** satisfies the relevant property of being zero when applied to a constant, and, in general, to any power function of non-negative integer degree less than m, if its order μ is such that $m-1 < \mu \le m$.

Gorenflo and Mainardi (1997) have shown the essential relationships between the two fractional derivatives (when both of them exist),

$$
{}_tD_*^\mu\, f(t) = \begin{cases} {}_tD^\mu \left[f(t) - \displaystyle\sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^k}{k!} \right], \\ {}_tD^\mu\, f(t) - \displaystyle\sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)\, t^{k-\mu}}{\Gamma(k-\mu+1)}, \end{cases} \qquad m-1 < \mu < m\,. \quad (A.8)
$$

In particular, if $m = 1$ we have

$$
{}_tD_*^\mu\, f(t) = \begin{cases} {}_tD^\mu \left[f(t) - f(0^+) \right], \\ {}_tD^\mu\, f(t) - \dfrac{f(0^+)\, t^{-\mu}}{\Gamma(1-\mu)}, \end{cases} \qquad 0 < \mu < 1\,. \qquad (A.9)
$$

The **Caputo fractional derivative**, represents a sort of regularization in the time origin for the **Riemann-Liouville fractional derivative**. We note that for its existence all the limiting values $f^{(k)}(0^+) := \lim_{t\to 0^+} f^{(k)}(t)$ are required to be finite for $k = 0, 1, 2. \ldots m-1$.

We observe the different behaviour of the two fractional derivatives at the end points of the interval $(m-1, m)$ namely when the order is any positive integer: whereas ${}_tD^{\mu}$ is, with respect to its order μ, an operator continuous at any positive integer, ${}_tD_*^{\mu}$ is an operator left-continuous since

$$\begin{cases} \displaystyle\lim_{\mu\to(m-1)^+} {}_tD_*^{\mu}\, f(t) \;=\; f^{(m-1)}(t) - f^{(m-1)}(0^+)\,, \\ \displaystyle\lim_{\mu\to m^-} {}_tD_*^{\mu}\, f(t) \;=\; f^{(m)}(t)\,. \end{cases} \tag{A.10}$$

We also note for $m-1 < \mu \le m$,

$${}_tD^{\mu}\, f(t) \;=\; {}_tD^{\mu}\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\mu-j}\,, \tag{A.11}$$

$${}_tD_*^{\mu}\, f(t) \;=\; {}_tD_*^{\mu}\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}\,. \tag{A.12}$$

In these formulae the coefficients c_j are arbitrary constants.

We point out the major utility of the Caputo fractional derivative in treating initial-value problems for physical and engineering applications where initial conditions are usually expressed in terms of integer-order derivatives. This can be easily seen using the **Laplace transformation**.

Writing the Laplace transform of a sufficiently well-behaved function $f(t)$ $(t \geq 0)$ as

$$\mathcal{L}\{f(t); s\} = \widetilde{f}(s) := \int_0^\infty \mathrm{e}^{-st} f(t)\, dt\,,$$

the known rule for the ordinary derivative of integer order $m \in \mathbf{N}$ is

$$\mathcal{L}\{{}_tD^m\, f(t); s\} = s^m\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k}\, f^{(k)}(0^+)\,, \quad m \in \mathbf{N}\,,$$

where

$$f^{(k)}(0^+) := \lim_{t\to 0^+}\, {}_tD^k f(t)\,.$$

For the **Caputo derivative** of order $\mu \in (m-1, m]$ $(m \in \mathbf{N})$ we have

$$\begin{aligned} &\mathcal{L}\{\,{}_tD_*^{\mu}\, f(t); s\} = s^{\mu}\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{\mu-1-k}\, f^{(k)}(0^+)\,, \\ &f^{(k)}(0^+) := \lim_{t\to 0^+}\ {}_tD^k f(t)\,. \end{aligned} \tag{A.13}$$

The corresponding rule for the **Riemann-Liouvile derivative** of order μ is

$$\begin{aligned} &\mathcal{L}\{\,{}_tD_t^{\mu}\, f(t); s\} = s^{\mu}\, \widetilde{f}(s) - \sum_{k=0}^{m-1} s^{m-1-k}\, g^{(k)}(0^+)\,, \\ &g^{(k)}(0^+) := \lim_{t\to 0^+}\ {}_tD^k g(t)\,, \quad g(t) := {}_tJ^{m-\mu}\, f(t)\,. \end{aligned} \tag{A.14}$$

Thus the rule (A.14) is more cumbersome to be used than (A.13) since it requires initial values concerning an extra function $g(t)$ related to the given $f(t)$ through a fractional integral.

However, when all the limiting values $f^{(k)}(0^+)$ are finite and the order is not integer, we can prove by that all $g^{(k)}(0^+)$ vanish so that the formula (A.14) simplifies into

$$\mathcal{L}\{\,{}_tD^{\mu}\,f(t);s\} = s^{\mu}\,\widetilde{f}(s)\,, \quad m-1<\mu<m\,. \tag{A.15}$$

For this proof it is sufficient to apply the Laplace transform to Eq. (A.8), by recalling that

$$\mathcal{L}\{t^{\nu};s\} = \Gamma(\nu+1)/s^{\nu+1}\,, \quad \nu>-1\,, \tag{A.16}$$

and then to compare (A.13) with (A.14).

For more details on the theory and applications of fractional calculus we recommend to consult in addition to the well-known books by Samko, Kilbas & Marichev (1993), by Miller & Ross (1993), by Podlubny (1999), those appeared in the last few years, by Kilbas, Srivastava & Trujillo (2006), by West, Bologna & Grigolini (2003), and by Zaslavsky (2005).

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The Standard Relaxation

The classical phenomenon of relaxation in its simplest form is known to be governed by a linear ordinary differential equation of order one, possibly non-homogeneous, that hereafter we recall with the corresponding solution. Denoting by $t \ge 0$ the time variable, $u = u(t)$ the field variable, and by ${}_tD^1$ the first-order time derivative, the **relaxation** differential equation (of homogeneous type) reads

$$ {}_tD^1 u(t) = -\lambda\, u(t)\,, \quad t \ge 0\,, \tag{2.1}$$

where λ is a positive constant denoting the inverse of some characteristic time.

The solution of (2.1), under the initial condition $u(0^+) = 1$, is called the **fundamental solution** and reads

$$ u(t) = \mathrm{e}^{-\lambda t}\,, \quad t \ge 0\,. \tag{2.2}$$

The Fractional Relaxation of Single Order

From the view-point of the **Fractional Calculus**, see for a short review the Appendix, there appear in the literature two ways of generalizing the equation (2.1), one way using the R-L, the other using the C fractional derivative. Adopting the notation of the Appendix for the R-L and C derivatives, see Eqs. (A.5) and (A.7) respectively, and denoting by β_1 the common fractional order, the two forms read respectively for $t \geq 0$

$$ {}_tD^1 \, u(t) = -\lambda \, {}_tD^{1-\beta} \, u(t) \,, \quad 0 < \beta \leq 1, \tag{2.3} $$

and

$$ {}_tD_*^\beta \, u(t) = -\lambda \, u(t) \,, \quad 0 < \beta \leq 1, \tag{2.4} $$

where now the positive constant λ has dimensions $[t]^{-\beta}$.

If we assume the same initial condition, *e.g.* $u(0^+) = 1$, it is not difficult to show the equivalence of the two forms by playing with the operators of standard and fractional integration and differentiation.

Both Eqs (2.3)-(2.4) are equivalent to the Volterra integral equation (of fractional type)

$$u(t) = u(0^+) - \lambda \, {}_tJ^\beta \, u(t) \, .$$

For example, we derive the R-L equation (2.3) from the fractional integral equation simply differentiating both sides of the latter, whereas we derive the fractional integral equation from the C equation (2.4) by fractional integration of order β. In fact, in view of the semigroup property (A.2) of the fractional integral, we note that

$${}_tJ^\beta \, {}_tD_*^\beta u(t) = {}_tJ^\beta \, {}_tJ^{1-\beta} \, {}_tD^1 \, u(t) = {}_tJ^1 \, {}_tD^1 \, u(t) = u(t) - u(0^+) \, .$$

In the limit $\beta = 1$ we recover the relaxation equation (2.1) with the solution (2.2).

The reader interested to have more details on the two forms of fractional relaxation may consult, for the R-L approach, the papers by Hilfer (2000), Metzler and Nonnenmacher (1995), whereas for the C approach the papers by Caputo and Mainardi (1971), Mainardi (1996), Gorenflo and Mainardi (1997).

By applying in Eqs. (2.3)-(2.4) the technique of the Laplace transforms for fractional derivatives of C and R-L type, see (A.13)-(A.15), we get the same result for the fundamental solution, namely

$$\widetilde{u}(s) = \frac{s^{\beta-1}}{s^\beta + \lambda}\,, \tag{2.5}$$

that, with the **Mittag-Leffler function** E_β, yields in the time domain

$$u(t) = E_\beta(-\lambda t^\beta)\,, \quad 0 < \beta \le 1\,. \tag{2.6}$$

Let us recall that the Mittag-Leffler function $E_\beta(z)$ ($\beta > 0$) is an entire transcendental function of order $1/\beta$, defined in the complex plane by the power series

$$E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta\, k + 1)}\,, \quad \beta > 0\,, \quad z \in \mathbf{C}\,.$$

We agree to refer to the equation (2.3) or (2.4) as the **simple fractional relaxation equation** in the R-L or C sense, respectively.

The spectral function of the Mittag-Leffler function

We remark that for $t \geq 0$ the function $E_\beta(-\lambda t^\beta)$ preserves the **complete monotonicity** of the exponential $\exp(-\lambda t)$:

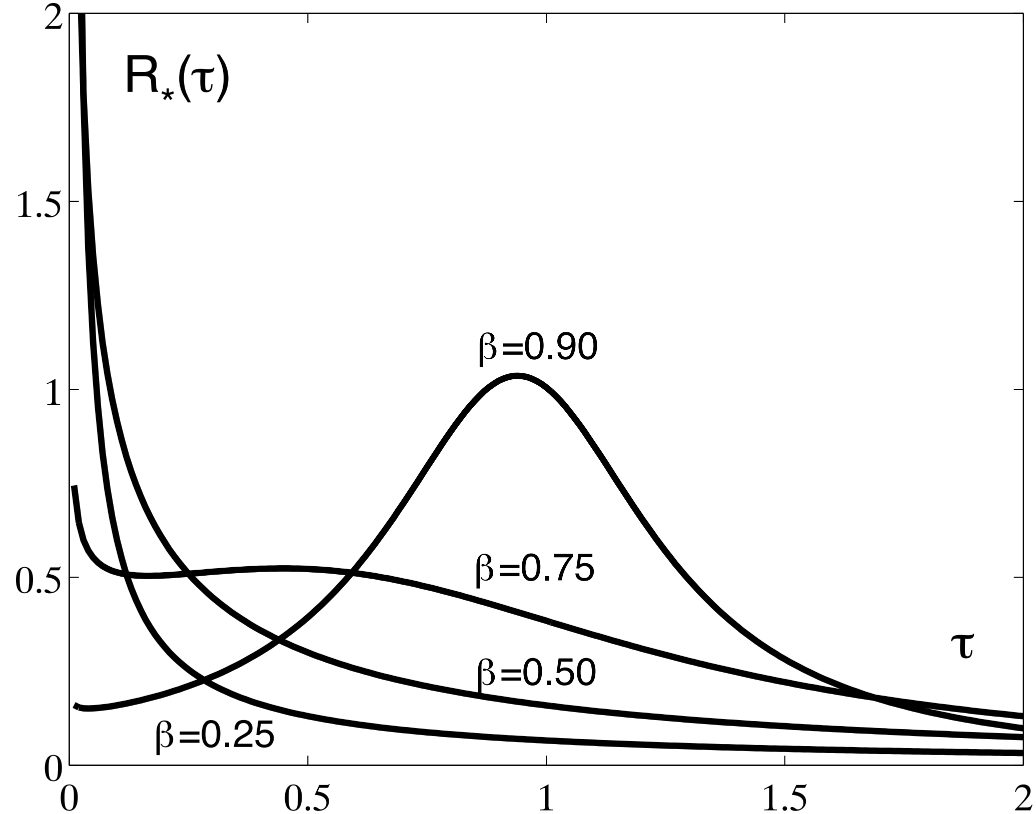
$$E_\beta(-\lambda t^\beta) = \frac{1}{\pi} \int_0^\infty \frac{\mathrm{e}^{-rt}}{r} \, \frac{\lambda r^\beta \, \sin(\beta\pi)}{\lambda^2 + 2\lambda \, r^\beta \, \cos(\beta\pi) + r^{2\beta}} \, dr \,, \quad 0 < \beta < 1 \,.$$

The spectral function is indeed non negative. Taking $\lambda = 1$ and using as variable the relaxation time $\tau = 1/r$ we have

$$E_\beta(-t^\beta) = \int_0^\infty \mathrm{e}^{-t/\tau} \, R_*(\tau) \, d\tau \,,$$

with

$$R_*(\tau) = \frac{1}{\pi\tau} \, \frac{\tau^\beta \, \sin(\beta\pi)}{1 + 2\lambda \, \tau^\beta \, \cos(\beta\pi) + \tau^{2\beta}} \, \geq 0 \,.$$



The Spectral Function

Asymptotic behaviour of the Mittag-Leffler function

We point out that the Mittag-Leffler function $E_\beta(-\lambda t^\beta)$ decreases at $t \to \infty$ no longer exponentially if $\beta < 1$ but like a power with exponent $-\beta$:

$$E_\beta(-\lambda t^\beta) \sim t^{-\beta}/[\lambda\Gamma(1-\beta)]\,.$$

If $\beta = 1/2$ we have for $t \geq 0$:

$$E_{1/2}(-\lambda\sqrt{t}) = \mathrm{e}^{\lambda^2 t}\,\mathrm{erfc}(\lambda\sqrt{t}) \sim 1/(\lambda\sqrt{\pi\, t})\,,\; t \to \infty\,,$$

where erfc denotes the **complementary error** function.

Plots of the Mittag-Leffler functions

In the following we show the solution (2.6) for a few values of the order $\beta = \beta_1$, $\beta_1 = 1/4, 1/2, 3/4, 1$, by assuming $\lambda = 1$.

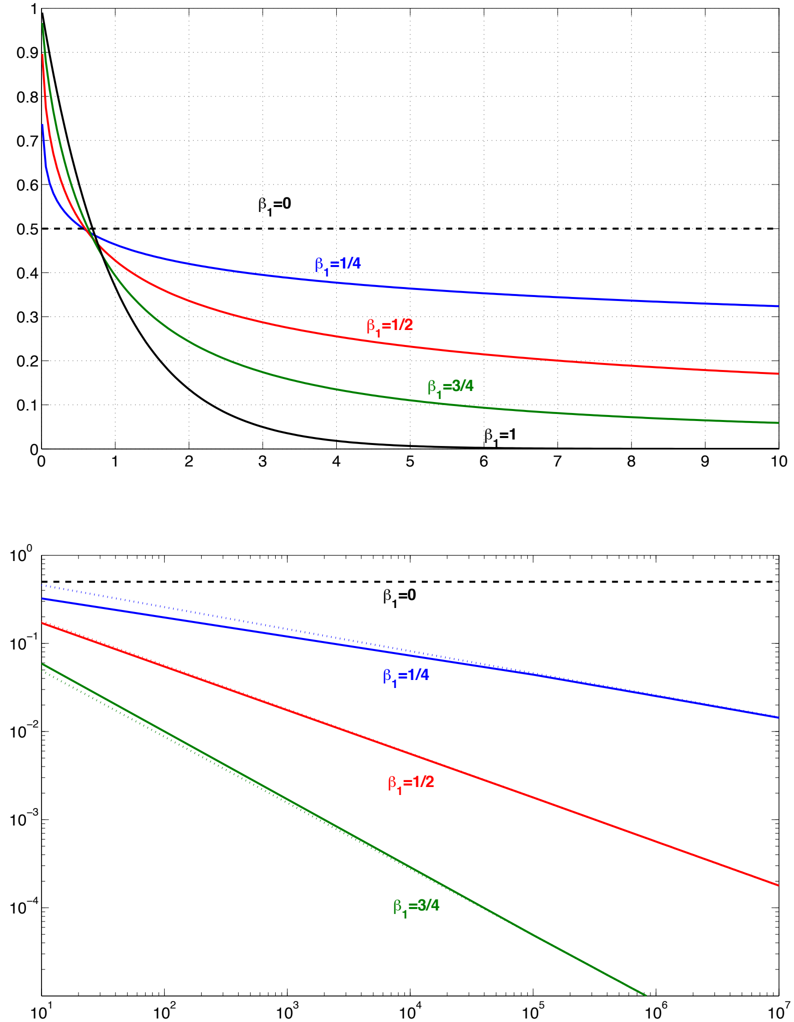
In the top plate for the time interval $[0, 10]$ (linear scales), and in the bottom plate for the time interval $[10^1, 10^7]$ (logarithmic scales). In the bottom plate we have added in dotted lines the asymptotic values for $t \to \infty$ in order to better visualize the power-law decay expressed by $t^{-\beta_1}/\Gamma(1-\beta_1)$ for the cases $0 < \beta_1 < 1$, whereas the case $\beta_1 = 1$ is not visible in view of the faster exponential decay.

In both plates we have shown in dashed line the singular solution for the limiting case $\beta_1 = 0$, stretching the definition of the Mittag-Leffler function to the geometric series,

$$E_0(z) = 1/(1-z)\,,$$

so

$$u(t) = \begin{cases} E_0(0) = 1\,, & t = 0\,, \\ E_0(-t^0) \equiv E_0(-1) = 1/2\,, & t > 0\,, \end{cases} \tag{2.7}$$



Fundamental solutions of the fractional relaxation of a single order $\beta_1 = 1/4, 1/2, 3/4, 1$. Top: linear scales; Bottom: logarithmic scales.

Fractional Relaxation of Distributed Order

The two forms for fractional relaxation

The simple fractional relaxation equations (2.3)-(2.4) can be generalized by using the notion **fractional derivative of distributed order**. We thus consider the so-called **fractional relaxation equation of distributed order**, in the two alternative forms involving the R-L and the C derivatives, that we write respectively as

$$ {}_tD^1 u(t) \,=\, -\lambda \int_0^1 p(\beta)\, {}_tD^{1-\beta} u(t)\, d\beta\,, \tag{3.1}$$

$$ \int_0^1 p(\beta)\, {}_tD_*^{\beta} u_*(t)\, d\beta \,=\, -\lambda\, u_*(t)\,, \tag{3.2}$$

subjected to the initial condition $u(0^+) = u_*(0^+) = 1$, where

$$ p(\beta) \geq 0\,, \quad \text{and} \quad \int_0^1 p(\beta)\, d\beta = c > 0\,. \tag{3.3}$$

The positive constant c can be taken as 1 if we want the integral to be normalized.

Clearly, some special conditions of regularity and behaviour near the boundaries will be required for the weight function $p(\beta)$. We conveniently require that its primitive $P(\beta) = \int_0^\beta p(\beta')\, d\beta'$ vanishes at $\beta = 0$ and is there continuous from the right, attains the value c at $\beta = 1$ and has at most finitely many (upwards) jump points in the half-open interval $0 < \beta \leq 1$, these jump points allowing delta contributions to $p(\beta)$ (particularly relevant for discrete distributions of orders).

Since for distributed order the solution depends on the selected approach (as we shall show hereafter), we now distinguish the fractional equations (3.1) and (3.2) and their fundamental solutions by decorating in the Caputo case the variable $u(t)$ with subscript $*$.

The present analysis is based on the application of the Laplace transformation with particular attention to some special cases. Here, for these cases, we shall provide plots of the corresponding solutions.

The integral formula for the fundamental solutions

Let us now apply the Laplace transform to Eqs. (3.1)-(3.2) by using the rules (A.15) and (A.13) appropriate to the R-L and C derivatives, respectively. Introducing the relevant functions

$$A(s) = s \int_0^1 p(\beta)\, s^{-\beta}\, d\beta\,, \tag{3.4}$$

and

$$B(s) = \int_0^1 p(\beta)\, s^{\beta}\, d\beta\,, \tag{3.5}$$

we then get for the R-L and C cases, after simple manipulation, the Laplace transforms of the corresponding fundamental solutions:

$$\widetilde{u}(s) = \frac{1}{s + \lambda A(s)}\,, \tag{3.6}$$

and

$$\widetilde{u}_*(s) = \frac{B(s)/s}{\lambda + B(s)}\,. \tag{3.7}$$

We easily note that in the particular case of a single order β_1

$$p(\beta) = \delta(\beta - \beta_1)$$

we have in (3.4):

$$A(s) = s^{1-\beta_1}\,,$$

and in (3.5):

$$B(s) = s^{\beta_1}\,,$$

Then, Eqs. (3.6) and (3.7) provide the same result (2.5) of the simple fractional relaxation, that is

$$\widetilde{u}(s) = \widetilde{u}_*(s) = \frac{s^{\beta-1}}{s^\beta + \lambda}\,, \qquad (2.5')$$

By inverting the Laplace transforms in (3.6) and (3.7) we obtain the fundamental solutions for the R-L and C fractional relaxation of distributed order.

Let us start with the R-L derivatives. We get (in virtue of the Titchmarsh theorem on Laplace inversion) the representation

$$u(t) = -\frac{1}{\pi} \int_0^\infty \mathrm{e}^{-rt} \, \mathrm{Im} \left\{ \widetilde{u} \left(r\mathrm{e}^{i\pi} \right) \right\} \, dr \,, \tag{3.8}$$

that requires the expression of $-\mathrm{Im}\left\{1/[s + \lambda A(s)]\right\}$ along the ray $s = r\,\mathrm{e}^{i\pi}$ with $r > 0$ (the branch cut of the function $s^{-\beta}$). We write

$$A\left(r\,\mathrm{e}^{i\pi}\right) = \rho \, \cos(\pi\gamma) + i\rho \sin(\pi\gamma) \,, \tag{3.9}$$

$$\begin{cases} \rho = \rho(r) = \left| A\left(r\,\mathrm{e}^{i\pi}\right)\right| \,, \\ \gamma = \gamma(r) = \dfrac{1}{\pi} \arg\left[A\left(r\,\mathrm{e}^{i\pi}\right)\right] . \end{cases} \tag{3.10}$$

Then, after simple calculations, we get

$$u(t) = \int_0^\infty \mathrm{e}^{-rt} \, H(r;\lambda) \, dr \,, \tag{3.11}$$

$$H(r;\lambda) = \frac{1}{\pi} \, \frac{\lambda\,\rho\,\sin(\pi\gamma)}{r^2 - 2\lambda\, r\,\rho\,\cos(\pi\gamma) + \lambda^2\rho^2} \ge 0 \,. \tag{3.12}$$

Similarly for the C derivatives we obtain

$$u_*(t) = -\frac{1}{\pi} \int_0^\infty \mathrm{e}^{-rt} \, \mathrm{Im}\left\{\widetilde{u}_*\left(r\mathrm{e}^{i\pi}\right)\right\} \, dr \,, \tag{3.13}$$

that requires the expression of $-\mathrm{Im}\,\{B(s)/[s(\lambda + B(s))]\}$ along the ray $s = r\,\mathrm{e}^{i\pi}$ with $r > 0$ (the branch cut of the function s^β). We write

$$B\left(r\,\mathrm{e}^{\,i\pi}\right) = \rho_* \, \cos(\pi\gamma_*) + i\rho_* \sin(\pi\gamma_*) \,, \tag{3.14}$$

$$\begin{cases} \rho_* = \rho_*(r) = \left|B\left(r\,\mathrm{e}^{i\pi}\right)\right| \,, \\ \gamma_* = \gamma_*(r) = \dfrac{1}{\pi} \arg\left[B\left(r\,\mathrm{e}^{i\pi}\right)\right] \,. \end{cases} \tag{3.15}$$

After simple calculations we get

$$u_*(t) = \int_0^\infty \mathrm{e}^{\,-rt} \, K(r;\lambda) \, dr \,, \tag{3.16}$$

$$K(r;\lambda) = \frac{1}{\pi\, r} \, \frac{\lambda\rho_* \, \sin(\pi\gamma_*)}{\lambda^2 + 2\lambda\, \rho_* \, \cos(\pi\gamma_*) + \rho_*^2} \geq 0 \,. \tag{3.17}$$

We note from (3.11) and (3.16) that, being $H(r;\lambda)$ and $K(r;\lambda)$ non-negative functions of r for any $\lambda \in \mathbf{R}^+$, the fundamental solutions $u(t)$ and $u_*(t)$ keep the relevant property to be *completely monotone*.

The integral expressions (3.11) and (3.16) provide a sort of spectral representation of the fundamental solutions that will be used to numerically evaluate these solutions in some examples considered as interesting cases.

Furthermore, it is quite instructive to compute for the fundamental solutions their asymptotic expressions for $t \to 0$ and $t \to \infty$ because they provide their analytical (even if approximated) representations for sufficiently short and long time respectively, and useful checks for the numerical evaluation in the above time ranges.

For deriving these asymptotic representations we shall apply the Tauberian theory of Laplace transforms. According to this theory the asymptotic behaviour of a function $f(t)$ near $t = \infty$ and $t = 0$ is (formally) obtained from the asymptotic behaviour of its Laplace transform $\widetilde{f}(s)$ for $s \to 0^+$ and for $s \to +\infty$, respectively. For this purpose we note the asymptotic representations,
from (3.6):

$$\widetilde{u}(s) \sim \begin{cases} \dfrac{1}{\lambda\, A(s)}\,, & s \to 0^+\,,\ \text{being } A(s)/s >> \lambda\,, \\ \dfrac{1}{s}\left[1 - \lambda\,\dfrac{A(s)}{s}\right], & s \to +\infty\,,\ \text{being } A(s)/s << 1/\lambda\,, \end{cases} \tag{3.18}$$

and from (3.7):

$$\widetilde{u}_*(s) \sim \begin{cases} \dfrac{1}{\lambda}\,\dfrac{B(s)}{s}\,, & s \to 0^+\,,\ \text{being } B(s) << \lambda\,, \\ \dfrac{1}{s}\left[1 - \dfrac{\lambda}{B(s)}\right], & s \to +\infty\,,\ \text{being } B(s) >> \lambda\,. \end{cases} \tag{3.19}$$

Fractional relaxation of distributed order: examples

Since finding explicit solution formulas is not possible for the relaxation equations (3.1) and (3.2) we shall concentrate our interest to some typical choices for the weight function $p(\beta)$ in (3.3) that characterizes the order distribution.

For these choices we present the numerical evaluation of the Titchmarsh integral formula, see Eqs (3.8)-(3.12) for $u(t)$ (the R-L case), and Eqs. (3.13)-(3.17) for $u_*(t)$ (the C case).

The numerical results are checked by verifying the matching with the asymptotic expressions of $u(t)$ and $u_*(t)$ as $t \to 0$ and $t \to +\infty$, obtained via the Tauberian theory for Laplace transforms., according to Eqs. (3.18)-(3.19).

The double-order fractional relaxation

We now consider the choice

$$p(\beta) = p_1 \delta(\beta - \beta_1) + p_2 \delta(\beta - \beta_2)\,, \quad 0 < \beta_1 < \beta_2 \le 1\,, \tag{4.1}$$

where the constants p_1 and p_2 are both positive, conveniently restricted to the normalization condition $p_1 + p_2 = 1$. Then for the R-L case we have

$$A(s) = p_1\, s^{1-\beta_1} + p_2\, s^{1-\beta_2}\,, \tag{4.2}$$

so that, inserting (4.2) in (3.6),

$$\widetilde{u}(s) = \frac{1}{s[1 + \lambda(p_1\, s^{-\beta_1} + p_2\, s^{-\beta_2})]}\,, \tag{4.3}$$

Similarly, for the C case we have

$$B(s) = p_1\, s^{\beta_1} + p_2\, s^{\beta_2}\,, \tag{4.3}$$

so that, inserting (4.3) in (3.7),

$$\widetilde{u}_*(s) = \frac{p_1\, s^{\beta_1} + p_2\, s^{\beta_2}}{s[\lambda + p_1\, s^{\beta_1} + p_2\, s^{\beta_2}]}\,. \tag{4.4}$$

We leave as an exercise the derivation of the spectral functions $H(r;\lambda)$ and $K(r;\lambda)$ of the corresponding fundamental solutions, that are used for the numerical computation. The numerical results are checked by their matching with the asymptotic expressions that we evaluate by invoking the Tauberian theory and using Eqs. (3.18)-(3.19) jointly with Eqs (4.2)-(4.3) respectively.

For the R-L-case we note that in (4.2) $s^{1-\beta_1}$ is negligibly small in comparison with $s^{1-\beta_2}$for $s \to 0^+$ and, viceversa, $s^{1-\beta_2}$ is negligibly small in comparison to $s^{1-\beta_1}$for $s \to +\infty$.

Similarly for the C-case we note that in (4.3) s^{β_2} is negligibly small in comparison to $s^{\beta_1}$for $s \to 0^+$ and, viceversa, s^{β_1} is negligibly small in comparison $s^{\beta_2}$for $s \to +\infty$.

As a consequence of these considerations we get for the R-L case, if $\beta_2 < 1$,

$$\widetilde{u}(s) \sim \begin{cases} \dfrac{1}{\lambda\, p_2}\, s^{\beta_2 - 1}\,, & s \to 0^+\,, \\ \dfrac{1}{s}\left(1 - \lambda\, p_1\, s^{-\beta_1}\right), & s \to +\infty\,, \end{cases} \tag{4.5}$$

so that

$$u(t) \sim \begin{cases} \dfrac{1}{\lambda\, p_2}\, \dfrac{t^{-\beta_2}}{\Gamma(1-\beta_2)}\,, & t \to +\infty\,, \\ 1 - \lambda\, p_1\, \dfrac{t^{\beta_1}}{\Gamma(1+\beta_1)}\,, & t \to 0^+\,. \end{cases} \tag{4.6}$$

We note that the Eq. (4.5a) and henceforth Eq. (4.6a) lose their meaning for $\beta_2 = 1$. In this case we need a more careful reasoning: we consider the expression for $s \to 0$ provided by (3.18) as it stands, that is

$$\widetilde{u}(s) \sim \frac{1}{\lambda\left[p_1 s^{1-\beta_1} + p_2\right]} = \frac{1}{\lambda\, p_1}\, \frac{1}{s^{1-\beta_1} + p_2/(\lambda\, p_1)}\,. \tag{4.7}$$

In virtue of the Laplace transform pair

$$t^{\nu-1} E_{\mu,\nu}\left(-q t^{\mu}\right) \div \frac{s^{\mu-\nu}}{s^{\mu}+q}\,, \tag{4.8}$$

see Eq. (1.80) in Podlubny (1999), where $E_{\mu,\nu}$ denotes the Mittag-Leffler function in two parameters
we get, with $q = p_2/(\lambda\, p_1)$ and $\mu = \nu = 1 - \beta_1$, as $t \to +\infty$:

$$u(t) \sim \frac{1}{\lambda\, p_1}\, t^{-\beta_1}\, E_{1-\beta_1,1-\beta_1}\left(-q t^{1-\beta_1}\right) = -\frac{1}{\lambda\, p_1}\,\frac{d}{dt} E_{1-\beta_1}\left(-q t^{1-\beta_1}\right). \tag{4.9}$$

Taking into account the asymptotic behaviour of the Mittag-Leffler function, we finally get

$$u(t) \sim \lambda\, \frac{p_1}{p_2}\, \frac{1-\beta_1}{\Gamma(\beta_1)}\, t^{-(2-\beta_1)} \quad \text{as} \quad t \to +\infty\,. \tag{4.10}$$

The generalized Mittag-Leffler function

The Mittag-Leffler function $E_{\mu,\nu}(z)$ ($\Re\{\mu\} > 0,\ \nu \in \mathbf{C}$) is defined by the power series

$$E_{\mu,\nu}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu\, k + \nu)}\,, \quad z \in \mathbf{C}\,.$$

It generalizes the classical Mittag-Leffler function to which it reduces for $\nu = 1$. It is an entire transcendental function of order $1/\Re\{\mu\}$.

With $\mu, \nu \in \mathbf{R}$ the function $E_{\mu,\nu}(-x)$ ($x \geq 0$) turns a completely monotonic function of x if $0 < \mu \leq 1$ and $\nu \geq \mu$, see Miller-Samko (2001). This property is still valid when $x = q\, t^{\mu}$ ($q > 0$). In particular, for $0 < \mu = \nu < 1$ we note

$$q\, t^{-(1-\mu)}\, E_{\mu,\mu}\left(-q\, t^{\mu}\right) = -\frac{d}{dt} E_{\mu}\left(-q\, t^{\mu}\right) \sim -\frac{\mu}{q\,\Gamma(1-\mu)}\, t^{-(\mu+1)}\,, \quad t \to +\infty\,.$$

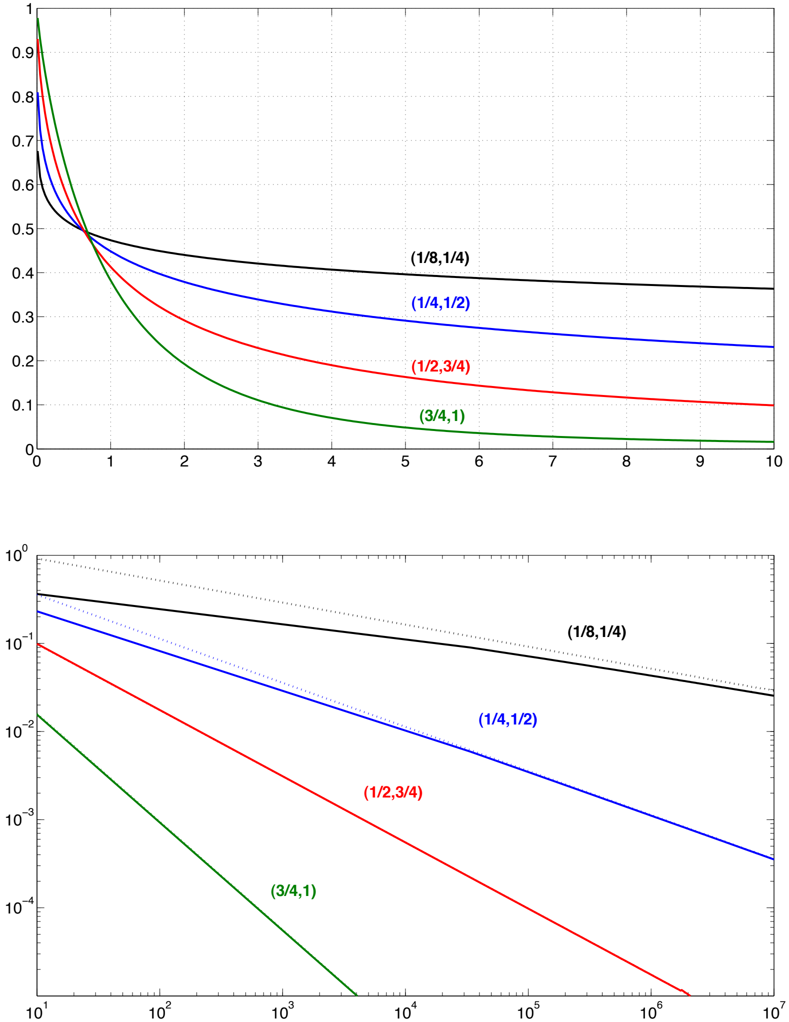
Similarly for the C case we get:

$$\widetilde{u}_*(s) \sim \begin{cases} \dfrac{p_1}{\lambda}\, s^{\beta_1 - 1}\,, & s \to 0^+\,, \\ \dfrac{1}{s}\left(1 - \dfrac{\lambda}{p_2} s^{-\beta_2}\right), & s \to +\infty\,, \end{cases} \tag{4.11}$$

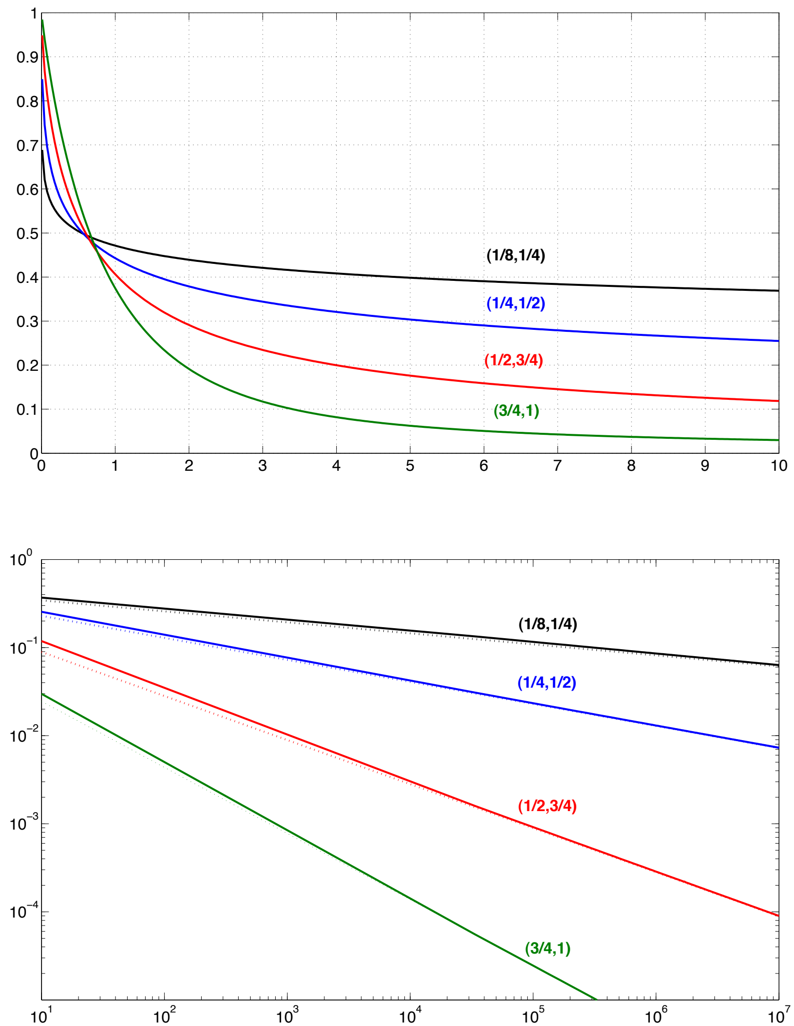
so that

$$u_*(t) \sim \begin{cases} \dfrac{p_1}{\lambda}\, \dfrac{t^{-\beta_1}}{\Gamma(1-\beta_1)}\,, & t \to +\infty\,, \\ 1 - \dfrac{\lambda}{p_2}\, \dfrac{t^{\beta_2}}{\Gamma(1+\beta_2)}\,, & t \to 0^+\,. \end{cases} \tag{4.12}$$

We exhibit in the following Figures the plots of the fundamental solutions for R-L and C fractional relaxation, respectively, in some $\{\beta_1, \beta_2\}$ combinations: $\{1/8, 1/4\}$; $\{1/4, 1/2\}$; $\{1/2, 3/4\}$; $\{3/4, 1\}$. We have chosen $p_1 = p_2 = 1/2$ and, as usual $\lambda = 1$. From the plots the reader is expected to verify the role played by the different orders for small and large times according to the corresponding asymptotic expressions, see Eqs. (4.6), (4.9)-(4.10) and (4.12).



Fundamental solutions of the R-L-fractional relaxation of double order in some $\{\beta_1, \beta_2\}$ combinations: $\{1/8, 1/4\}$; $\{1/4, 1/2\}$; $\{1/2, 3/4\}$; $\{3/4, 1\}$. Top: linear scales; Bottom: logarithmic scales.



Fundamental solutions of the C-fractional relaxation of double order in some $\{\beta_1, \beta_2\}$ combinations: $\{1/8, 1/4\}$; $\{1/4, 1/2\}$; $\{1/2, 3/4\}$; $\{3/4, 1\}$. Top: linear scales; Bottom: logarithmic scales.

The uniformly distributed order fractional relaxation

We now consider the choice

$$p(\beta) = 1\,, \quad 0 < \beta < 1\,. \tag{4.13}$$

For the R-L case we have

$$A(s) = s \int_0^1 s^{-\beta}\, d\beta = \frac{s-1}{\log s}\,, \tag{4.14}$$

hence, inserting (4.14) in (3.6)

$$\widetilde{u}(s) = \frac{\log s}{s \log s + \lambda\,(s-1)}\,. \tag{4.15}$$

For the C case we have

$$B(s) = \int_0^1 s^{\beta}\, d\beta = \frac{s-1}{\log s}\,, \tag{4.16}$$

hence, inserting (4.16) in (3.7),

$$\widetilde{u}_*(s) = \frac{1}{s}\frac{s-1}{\lambda \log s + s - 1} = \frac{1}{s} - \frac{1}{s}\frac{\lambda \log s}{\lambda \log s + s - 1}\,. \tag{4.17}$$

We note that for this special order distribution we have $A(s) = B(s)$ but the corresponding fundamental solutions are quite different, as we see from their Laplace transforms (4.15) and (4.17).

Then, invoking the Tauberian theory for **regularly varying functions** (power functions multiplied by **slowly varying functions**[1]), a topic adequately treated in the treatise on Probability by Feller (1971)Chapter XIII.5, we have the following asymptotic expressions for the R-L and C cases.

For the R-L case we get

$$\widetilde{u}(s) \sim \begin{cases} \dfrac{\log s}{\lambda(s-1)}\,, & s \to 0^+\,, \\ \dfrac{1}{s}\left[1 - \lambda\,\dfrac{s-1}{s\log s}\right], & s \to +\infty\,, \end{cases} \qquad (4.18)$$

[1]**Definition:** We call a (measurable) positive function $a(y)$, defined in a right neighbourhood of zero, **slowly varying at zero** if $a(cy)/a(y) \to 1$ with $y \to 0$ for every $c > 0$. We call a (measurable) positive function $b(y)$, defined in a neighbourhood of infinity, **slowly varying at infinity** if $b(cy)/b(y) \to 1$ with $y \to \infty$ for every $c > 0$. Examples: $(\log y)^\gamma$ with $\gamma \in \mathbf{R}$ and $\exp\,(\log y/\log\,\log y)$.

so

$$u(t) \sim \begin{cases} \dfrac{1}{\lambda}\,\mathrm{e}^{\,t}\,\mathcal{E}_1(t) \sim \dfrac{1}{\lambda\,t}\,, & t \to +\infty\,, \\ 1 - \dfrac{\lambda}{|\log{(1/t)}|}\,, & t \to 0^+\,. \end{cases} \tag{4.19}$$

In (4.19a) $\mathcal{E}_1(t) := \displaystyle\int_t^\infty \frac{\mathrm{e}^{-u}}{u}\,du$ denotes the exponential integral, see *e.g.* Abramowitz and Stegun (1965), Ch. 5 and the Laplace transform pair (29.3.100).

For the C case we get

$$\widetilde{u}_*(s) \sim \begin{cases} \dfrac{1}{\lambda\,s\log{(1/s)}}\,, & s \to 0^+\,, \\ \dfrac{1}{s} - \dfrac{\lambda\log s}{s^2}\,, & s \to +\infty\,, \end{cases} \tag{4.20}$$

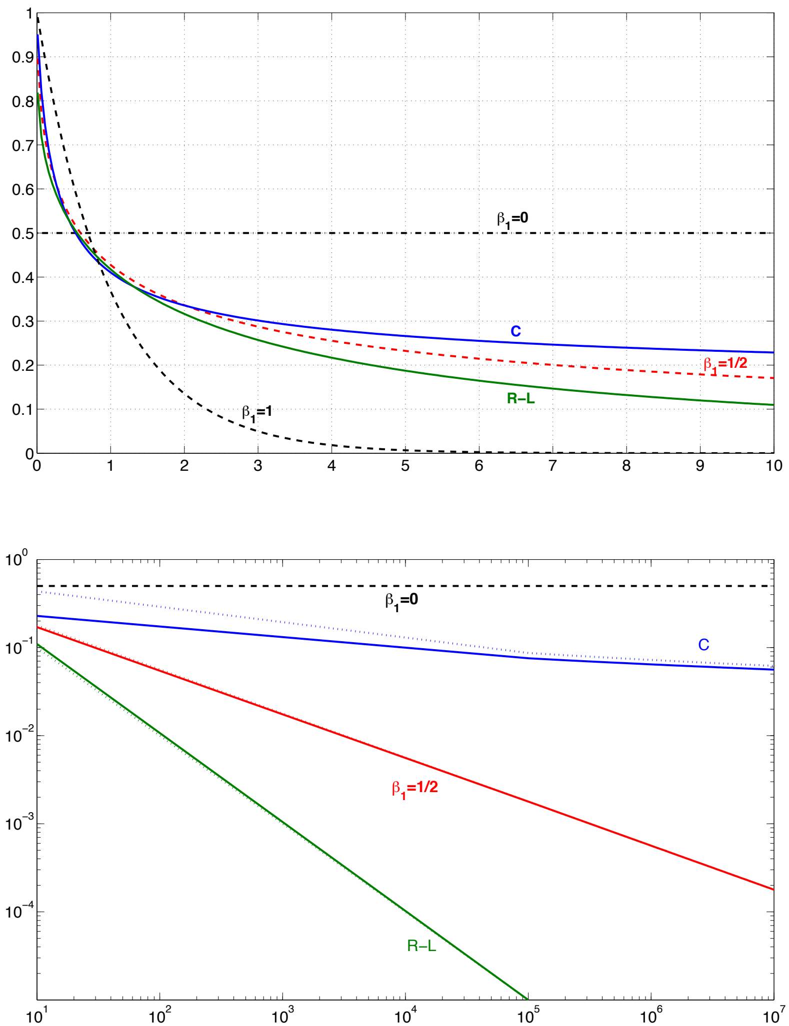
so

$$u_*(t) \sim \begin{cases} \dfrac{1}{\lambda\log t}\,, & t \to +\infty\,, \\ 1 - \lambda\,t\log{(1/t)}\,, & t \to 0^+\,. \end{cases} \tag{4.21}$$

In the next Figure we display the plots of the fundamental solutions for R-L and C uniformly distributed fractional relaxation, adopting as previously, in the top plate, linear scales ($0 \le t \le 10$), and in the bottom plate, logarithmic scales ($10^1 \le t \le 10^7$).

For comparison in the top plate the plots for single orders $\beta_1 = 0, 1/2, 1$ are shown. We note that for $1 < t < 10$ the R-L and C plots are close to that for $\beta_1 = 1/2$ from above and from below, respectively.

In the bottom plate (where the plot for β_1 is not visible because of its faster exponential decay) we have added in dotted lines the asymptotic solutions for large times. We recognize that the C plot is decaying much slower than any power law whereas the R-L plot is decaying as t^{-1}; this means that for large times these plots are the border lines for all the plots corresponding to single order relaxation with $\beta_1 \in (0, 1)$.



Fundamental solutions for R-L and C uniformly distributed fractional relaxation in comparison with some of them for single orders. Top: linear scales; Bottom: logarithmic scales.

Fractional Relaxation: Conclusions

We have investigated the relaxation equation with (discretely or continuously) distributed order of fractional derivatives both in the Riemann-Liouville and in the Caputo sense. Such equations can be seen as simple models of more general distributed order fractional evolution in a Banach space where the relaxation parameter λ is replaced by an operator A acting in this space. A relevant example is time-fractional diffusion where in the linear case the individual modes exhibit fractional relaxation.

Our interest is focused on structural properties of the solutions, in particular on asymptotic behaviour at small and large times. In both approaches we find that the smallest order of occurring fractional differentiation determines the behavior near infinity, but the largest order the behaviour near zero, in analogy to the special form of time-fractional diffusion explicitly governed by the distributed order derivative. We see that the two parameters β_1 and β_2 play opposite roles in our two cases (R-L) and (C).

Contents of the SECOND PART: Fractional Diffusion

- **The Standard Diffusion**
- **The Anomolaous Diffusion**
- **The Space Fractional Diffusion Equation and the Lévy stable distributions**
- **The Time Fractional Diffusion Equation and the M-Wright distributions**
- **The Space-Time Fractional Diffusion Equation**
- **The Time Fractional Diffusion Equations of Distributed Order**

The Standard Diffusion

The German physiologist Adolf Fick in 1855 published his famous diffusion equation :

$$\frac{\partial}{\partial t} u(x,t) \;=\; D\,\frac{\partial^2}{\partial x^2}\,u(x,t)\,,$$

where D is a positive real number called *diffusion coefficient*.

Bachelier derivation (1900)

Einstein derivation (1905)

Probability interpretation of the standard diffusion

$$t \geq 0, \quad u(x,t) \text{ probability desity function}$$

$$A \subseteq \mathbf{R} \quad \text{Prob}\,\{ \text{ The object is in } A \text{ at time } t \ \} = \int_A u(x,t)\,dx\,.$$

Green function :

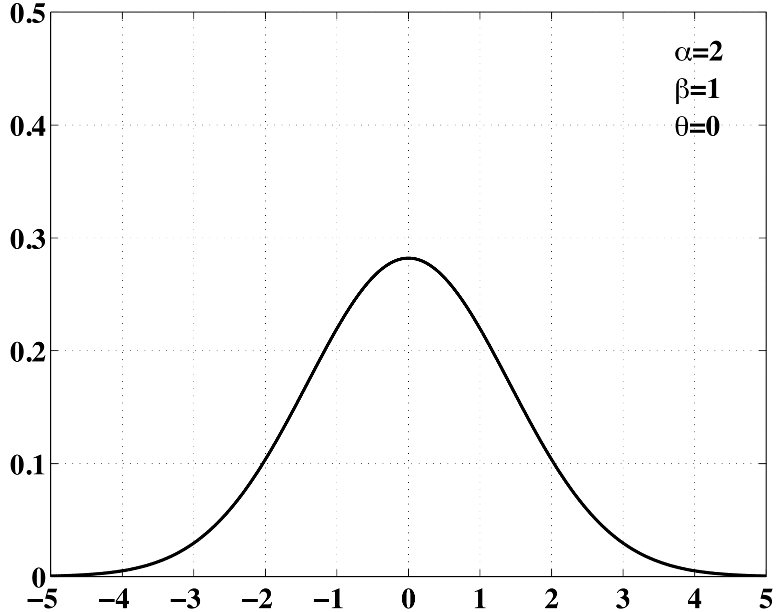
$$u(x,t) = \frac{1}{2\sqrt{\pi}}\,(Dt)^{-1/2}\,\mathrm{e}^{-\frac{x^2}{4Dt}} \qquad \text{for} \quad t > 0\,, \qquad u(x,0) = \delta(x)\,.$$

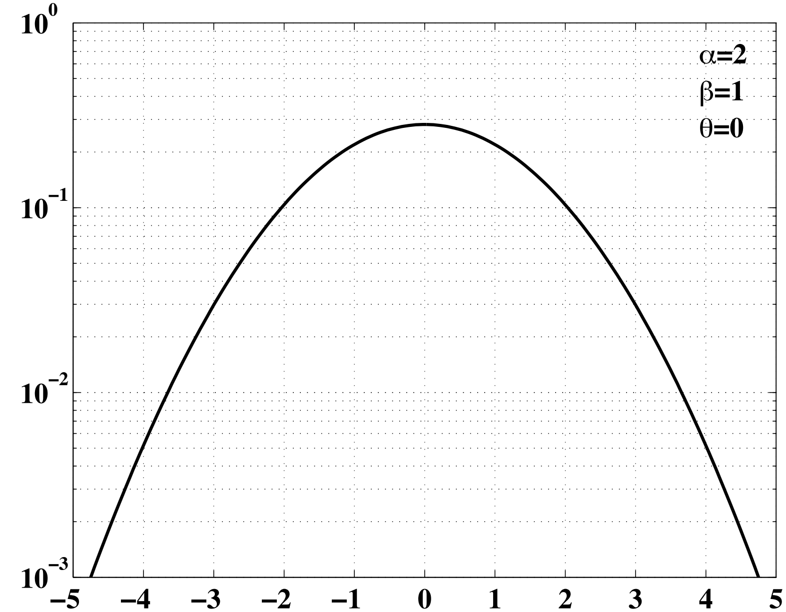
Variance :

$$Var(t) = \int_{-\infty}^{+\infty} x^2\,u(x,t)\ dx = \int_{-\infty}^{+\infty} \frac{x^2}{2\sqrt{\pi}}\,(Dt)^{-1/2}\,\mathrm{e}^{-\frac{x^2}{4Dt}}\ dx = 2Dt.$$

General initial condition :

$$u(x,t) = \rho(x)\,, \qquad u(x,t) = \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi}}\,(Dt)^{-1/2}\,\mathrm{e}^{-\frac{(x-x')^2}{4Dt}}\ \rho(x')\ dx'\,.$$





The Gaussian probability density

Karl Pearson's 'random walk' problem

A man starts from the point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after n stretches he is at a distance between r and $r + \delta r$ from his starting point O.
Nature, July 1905.

Details

- The problem was related to the way mosquitoes spread malaria.
- The mosquitoes' spread was well described by the diffusion equation.
- The positions of a number of mosquitoes starting from the same origin followed a Gaussian distribution with variance proportional to the elapsed time.

Random Walk = Brownian Motion $\longrightarrow$ Standard Diffusion

The Anomolous Diffusion

Examples of Anomalous Diffusion

- Charge carriers moving in amorphous media tend to get trapped by local imperfections and then released due to thermal fluctuations. $\longrightarrow$ random waiting times.

  H. Scher, E. Montroll, Anomalous transit-time dispersion in amorphous solids, **Physical Review B 12** (1975), pp. 2455-2477.

- Pollutants take longer times to travel than expected from classical diffusion, due to trapping caused by stagnant regions of zero velocity of the mean flow of the groundwater.

- Standard diffusion equation has to be replaced by the fractional version in order to take account of extremely long retention times of pollutants in a study on the effects of the system memory on contaminant patterns over long periods. M. Dentz, A. Cortis, H. Scher, B. Berkowitz, Time behaviour of solute transport in heterogeneous media: transition from anomalous to normal transport, **Advances in Water Resources 27** (2004), pp. 155-173.

- Protein molecules spend relatively long times trapped between nanometer-sized compartments in the actin cytoskeleton of the cell.

  K. Ritchie, X.-Y. Shan, J. Kondo, K. Iwasawa, T. Fujiwara and A. Kusumi, Detection of non-Brownian diffusion in the cell membrane in single molecule tracking, **Biophysical Journal 88**, (2005), pp. 2266-2277.

- Conformational fluctuations occur over a broad range of time scales and can be modeled by a generalized Langevin equation.

  W. Min, G. Luo, B. J. Cherayil, S. C. Kou, and X. S. Xie, Observation of a Power-Law Memory Kernel for Fluctuations within a Single Protein Molecule, **Physical Review Letters 94** 198302, (2005).

Broad range of waiting times $\longrightarrow$ Sub-diffusion

$$var(t) \approx t^{\beta}\,,\ 0 < \beta < 1$$

- Albatrosses fly at an approximately constant velocity for times having a very broad distribution before changing direction. This imply their patterns consist of long straight lines interrupted by localized random motion.

  G.M. Viswanathan, V. Afanasyev, S.V. Buldyrev, E.J. Murphy, P.A. Prince and H.E. Stanley, Levy Flight Search Patterns of Wandering Albatrosses, **Nature 381**, (2005), pp. 413-415.

- Spider monkeys move following a Lévy walk pattern.

  G. Ramos-Fernandez, J.L. Mateos, O. Miramontes, G. Cocho, H. Larralde and B. Ayala-Orozco, Levy walk patterns in the foraging movements of spider monkeys (Ateles geoffroyi), available at http://arxiv.org/abs/physics/0301019, (2003).

- Heavy particles perform long steps in the motion on the surface of a perfect crystal, since only turning angles obeying the symmetry of the crystal are allowed.

  J.M. Sancho, A.M. Lacasta, K. Lindenberg, I.M. Sokolov and A.H. Romero, Diffusion on a Solid Surface: Anomalous is Normal, **Physical Review Letters 92** 250601, (2004).

- Anomalous diffusion of tracer particles in a rapidly rotating annular tank.

  T.H. Solomon, E.R. Weeks and H.L. Swinney, Observation of anomalous diffusion and Lévy flights in a two-dimensional rotating flow, **Physical Review Letters 71**, 3975-3978, (1993).

Broad distribution of step lengths $\longrightarrow$ Super-diffusion

$$var(t) > 2Dt$$

The Space Fractional Diffusion Equation and the Lévy Stable distributions

Standard Diffusion Equation

$$\frac{\partial}{\partial t} u(x,t) \,=\, \frac{\partial^2}{\partial x^2}\, u(x,t)\,,$$

The Space-time Fractional Diffusion Equation

$${}_tD_*^{\beta}\, u(x,t) \,=\, {}_xD_\theta^{\alpha}\, u(x,t)\,, \quad u(x,0) = \delta(x)\,,$$

$$0 < \alpha \le 2\,, \quad |\theta| \le \min\{\alpha, 2-\alpha\}\,, \quad 0 < \beta \le 1\,.$$

where

${}_tD_*^{\beta}$	**Caputo fractional derivative**
${}_xD_\theta^{\alpha}$	**Riesz-Feller fractional derivative**

The Riesz-Feller fractional derivative

Writing the Fourier transform of a well-behaved function $f(x)$ $(x \in \mathbf{R})$

$$\mathcal{F}\{f(x);\kappa\} = \widehat{f}(\kappa) := \int_{-\infty}^{+\infty} \mathrm{e}^{+i\kappa x}\, f(x)\, dx\,,$$

we define the **Riesz-Feller** derivative:

$$\mathcal{F}\{\,{}_xD_\theta^\alpha\, f(x);\kappa\} = -\psi_\alpha^\theta(\kappa)\;\widehat{f}(\kappa)\,,$$

$$\psi_\alpha^\theta(\kappa) = |\kappa|^\alpha\, \mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2}\,, \quad 0 < \alpha \le 2\,, \quad |\theta| \le \min\{\alpha, 2-\alpha\}\,.$$

For $\alpha = 2$ (hence $\theta = 0$) we have $\mathcal{F}\{\,{}_xD_\theta^\alpha\, f(x);\kappa\} = -\kappa^2 = (-i\kappa)^2$, so we recover the standard second derivative. More generally for $\theta = 0$ we have $\mathcal{F}\{\,{}_xD_\theta^\alpha\, f(x);\kappa\} = -|\kappa|^\alpha = -(\kappa^2)^{\alpha/2}$ so

$$ {}_xD_0^\alpha = -\left(-\frac{d^2}{dx^2}\right)^{\alpha/2} . $$

In this case we refer to the LHS as simply to the **Riesz fractional derivative** of order α.

The **Riesz-Feller** derivative, in its explicit form, reads:

$$ \begin{aligned} {}_xD_\theta^\alpha\, f(x) &= \frac{\Gamma(1+\alpha)}{\pi} \left\{ \sin\left[(\alpha+\theta)\pi/2\right] \int_0^\infty \frac{f(x+\xi) - f(x)}{\xi^{1+\alpha}}\, d\xi \right. \\ &+ \left. \sin\left[(\alpha-\theta)\pi/2\right] \int_0^\infty \frac{f(x-\xi) - f(x)}{\xi^{1+\alpha}}\, d\xi \right\} . \end{aligned} $$

For $\alpha = 1$ and $\theta = \pm 1$ it reduces to

$$ {}_xD_{\pm 1}^1 = \pm\, {}_xD = \pm \frac{d}{dx} . $$

The Lévy stable densities

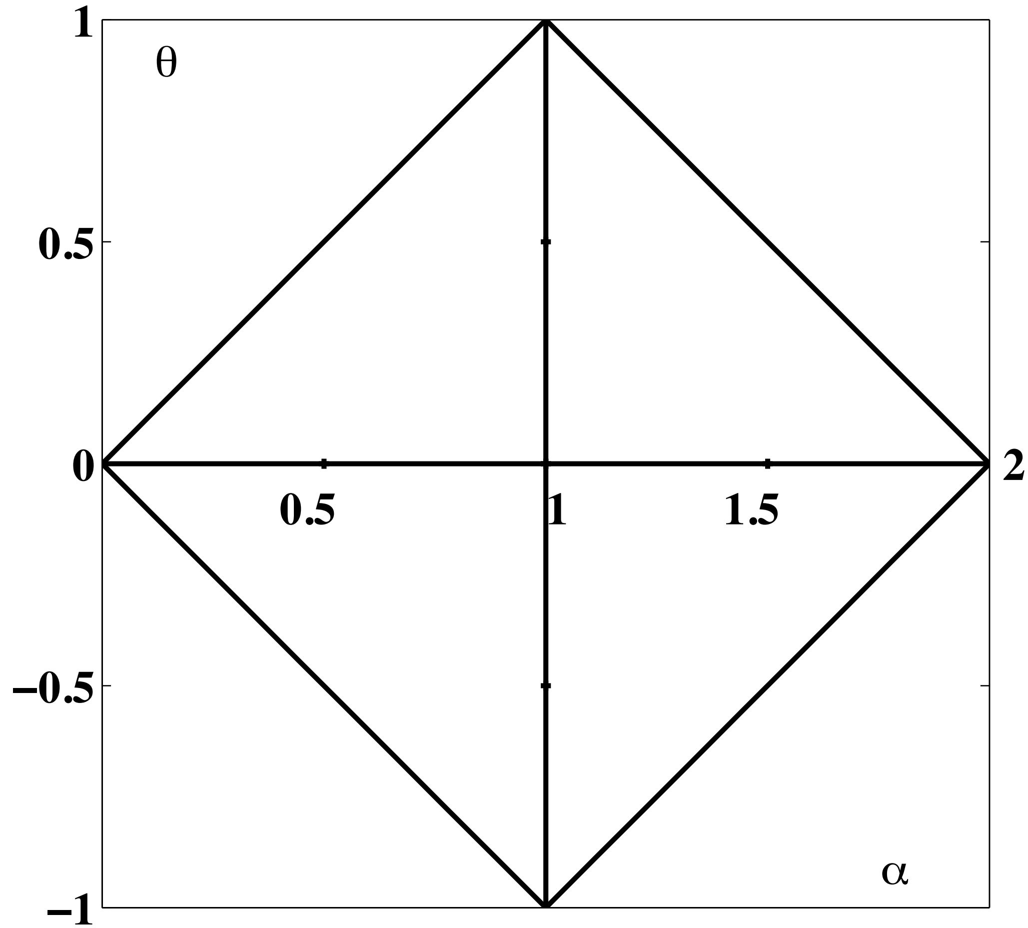
We recognize that the *Riesz-Feller* derivative is required to be the pseudo-differential operator[2] whose symbol $-\psi_\alpha^\theta(\kappa)$ is the logarithm of the characteristic function of a general *Lévy strictly stable* probability density with *index of stability* α and asymmetry parameter θ (improperly called *skewness*) according to Feller's parameterization, see Feller (1952), (1971), as revisited by Gorenflo and Mainardi (1998).

Indeed the Feller canonical form of the characteristic function the Lévy *strictly stable* densities $L_\alpha^\theta(x)$ (of order α and skewness θ) is

$$\widehat{L}_\alpha^\theta(\kappa) = \exp\left[-\psi_\alpha^\theta(\kappa)\right] , \quad \kappa \in \mathbf{R}$$

$$\psi_\alpha^\theta(\kappa) = |\kappa|^\alpha \, \mathrm{e}^{i(\operatorname{sign}\kappa)\theta\pi/2} , \quad 0 < \alpha \le 2 , \quad |\theta| \le \min\{\alpha, 2-\alpha\} .$$

[2]Let us recall that a generic pseudo-differential operator A, acting with respect to the variable $x \in \mathbf{R}$, is defined through its Fourier representation, namely $\int_{-\infty}^{+\infty} \mathrm{e}^{\,i\kappa x} \, A\,[f(x)]\, dx = \widehat{A}(\kappa)\, \widehat{f}(\kappa)$, where $\widehat{A}(\kappa)$ is referred to as symbol of A, given as $\widehat{A}(\kappa) = \left(A\,\mathrm{e}^{\,-i\kappa x}\right)\, \mathrm{e}^{\,+i\kappa x}$.



The Feller-Takayasu diamond

The Laplace-Fourier Representation and the Self-Similarity Property

In the Laplace-Fourier domain the Space-Time Fractional Diffusion Equation becomes:

$$s^{\beta}\,\widehat{\widetilde{u}}(\kappa,s) - s^{\beta-1} = -|\kappa|^{\alpha}\, i^{\theta\,\mathrm{sign}\,\kappa}\,\widehat{\widetilde{u}}(\kappa,s)\,,$$

hence

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{s^{\beta-1}}{s^{\beta}+|\kappa|^{\alpha}\, i^{\theta\,\mathrm{sign}\,\kappa}} \Longrightarrow u(x,t) = t^{-\beta/\alpha}\, K_{\alpha,\beta}^{\theta}\left(x/t^{\beta/\alpha}\right)\,.$$

where $K_{\alpha,\beta}^{\theta}$ is the **reduced Green function**.

The Mellin-Barnes Integral Representation

F. Mainardi, Yu. Luchko and G. Pagnini,

The fundamental solution of the space-time fractional diffusion equation,

Fractional Calculus and Applied Analysis 4 No 2 (2001), pp. 153-192.

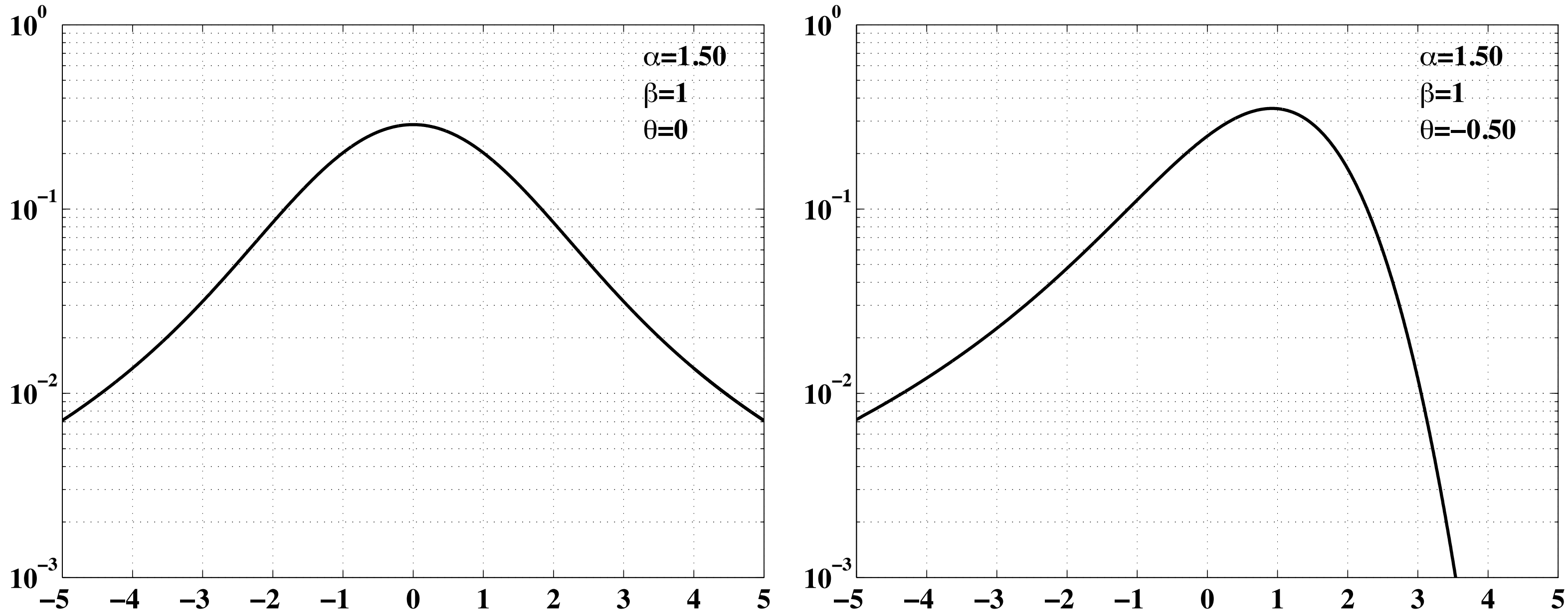
By setting

$$\rho = \frac{\alpha - \theta}{2\,\alpha}\,,$$

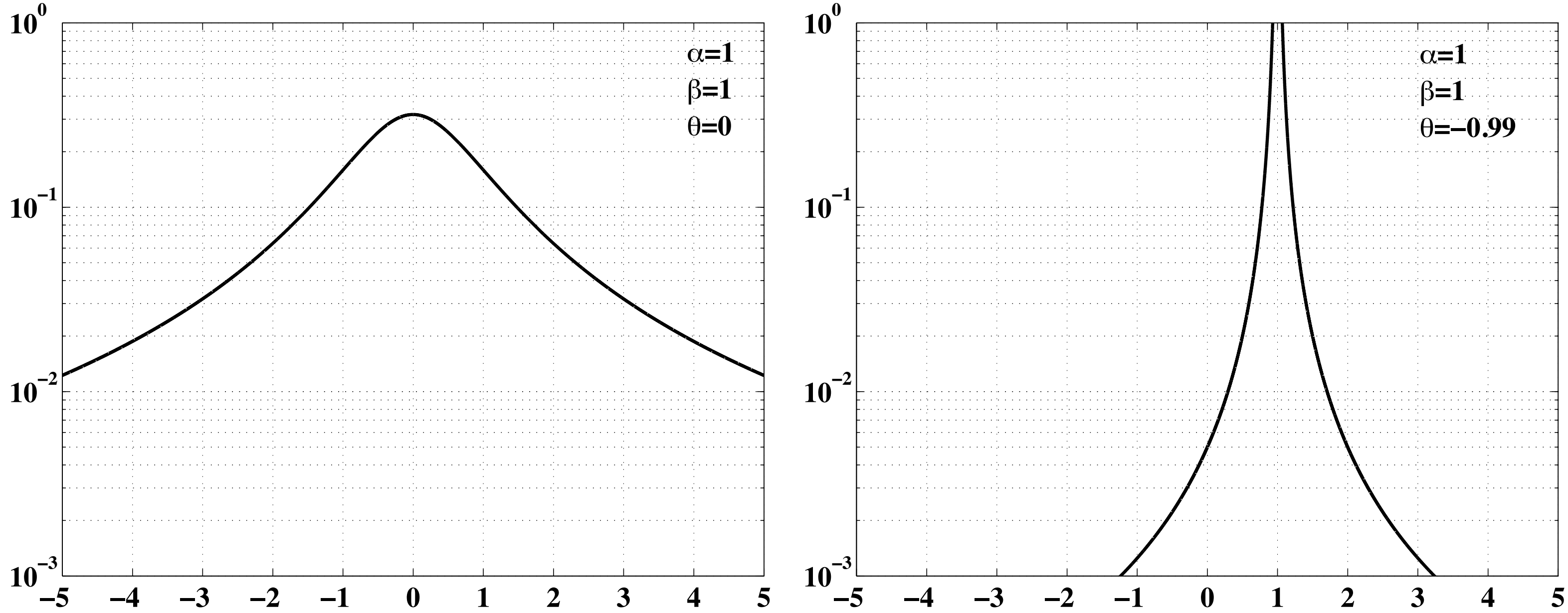
the Mellin-Barnes integral representation reads

$$K^{\theta}_{\alpha,\beta}(x) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(\frac{s}{\alpha})\,\Gamma(1-\frac{s}{\alpha})\,\Gamma(1-s)}{\Gamma(1-\frac{\beta}{\alpha}s)\,\Gamma(\rho\,s)\,\Gamma(1-\rho\,s)}\,x^{\,s}\,ds\,.$$

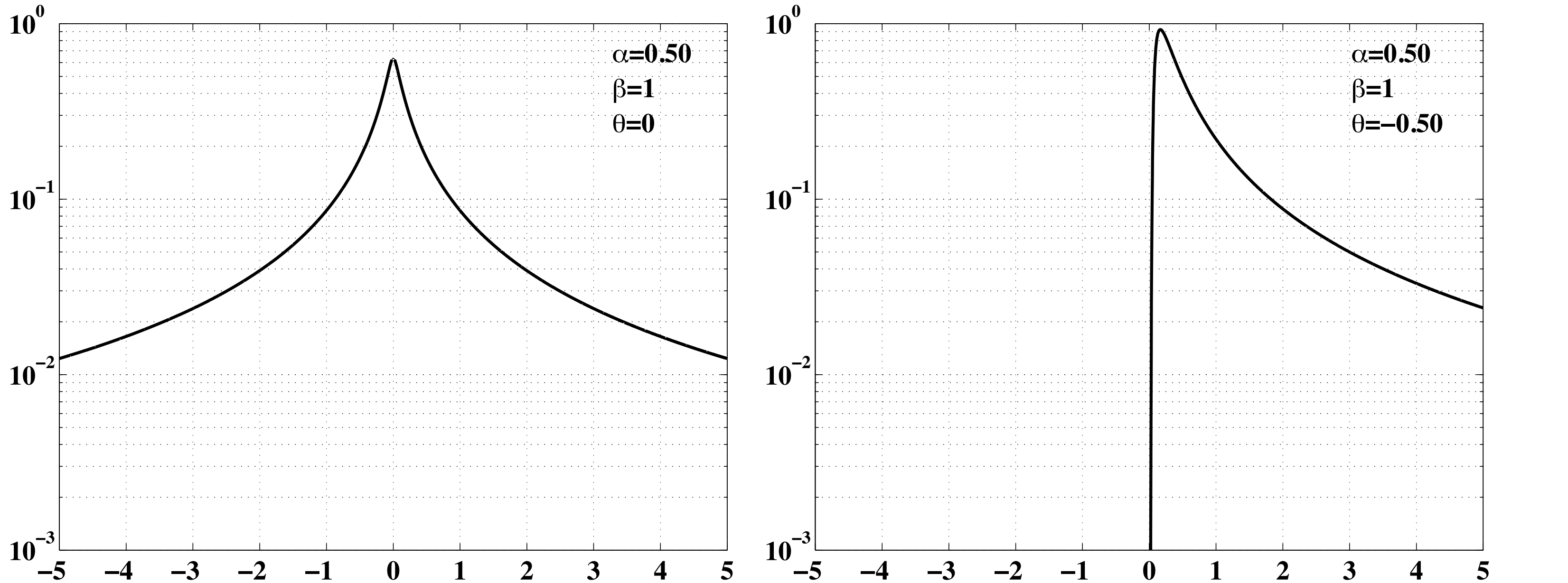
For $\beta = 1$ (**space-fractional diffusion**) $K^{\theta}_{\alpha,1}(x) := L^{\theta}_{\alpha}(x)$.
For $\alpha = 2$ (**time-fractional diffusion**) $K^{0}_{2,\beta}(x) := \frac{1}{2}M_{\beta/2}(x)$.
For $\alpha = \beta$ (**neutral-fractional diffusion**) $K^{\theta}_{\alpha,\alpha}(x) := N^{\theta}_{\alpha}(x)$.



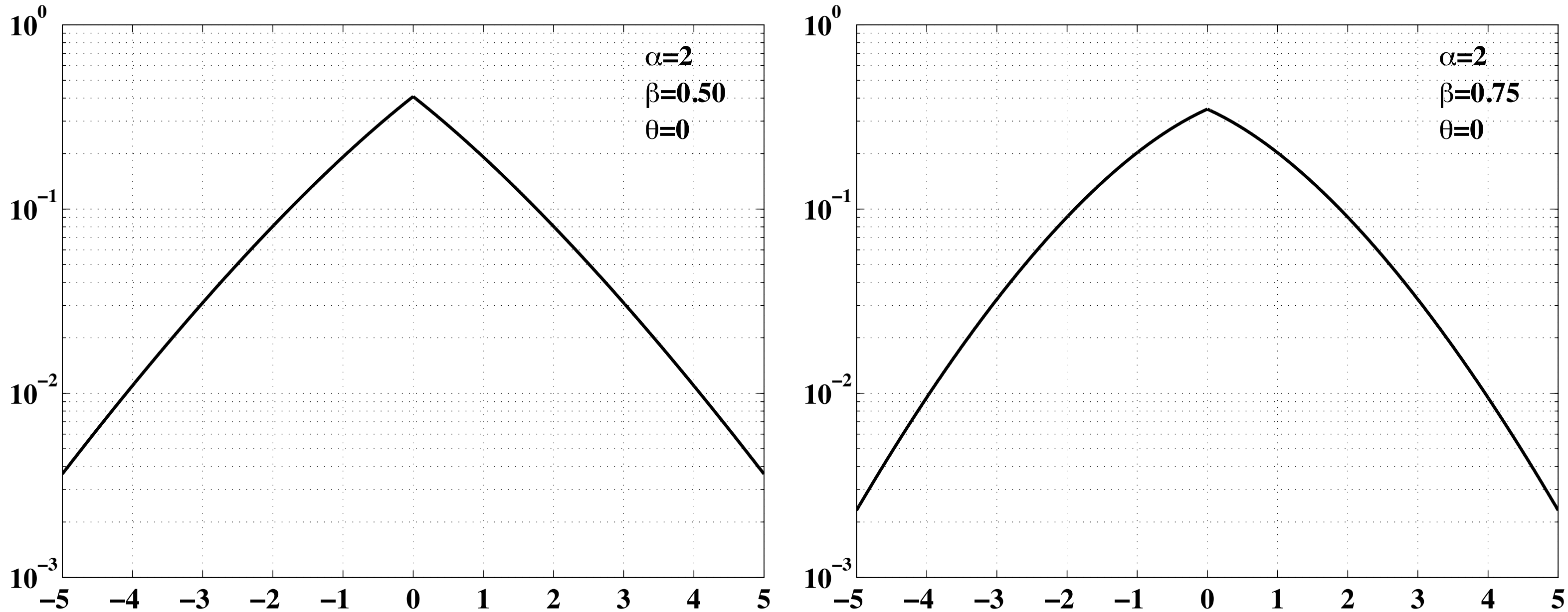
The Lévy probability density for $\alpha = 1.5$ and $\theta = 0$, $\theta = -0.50$



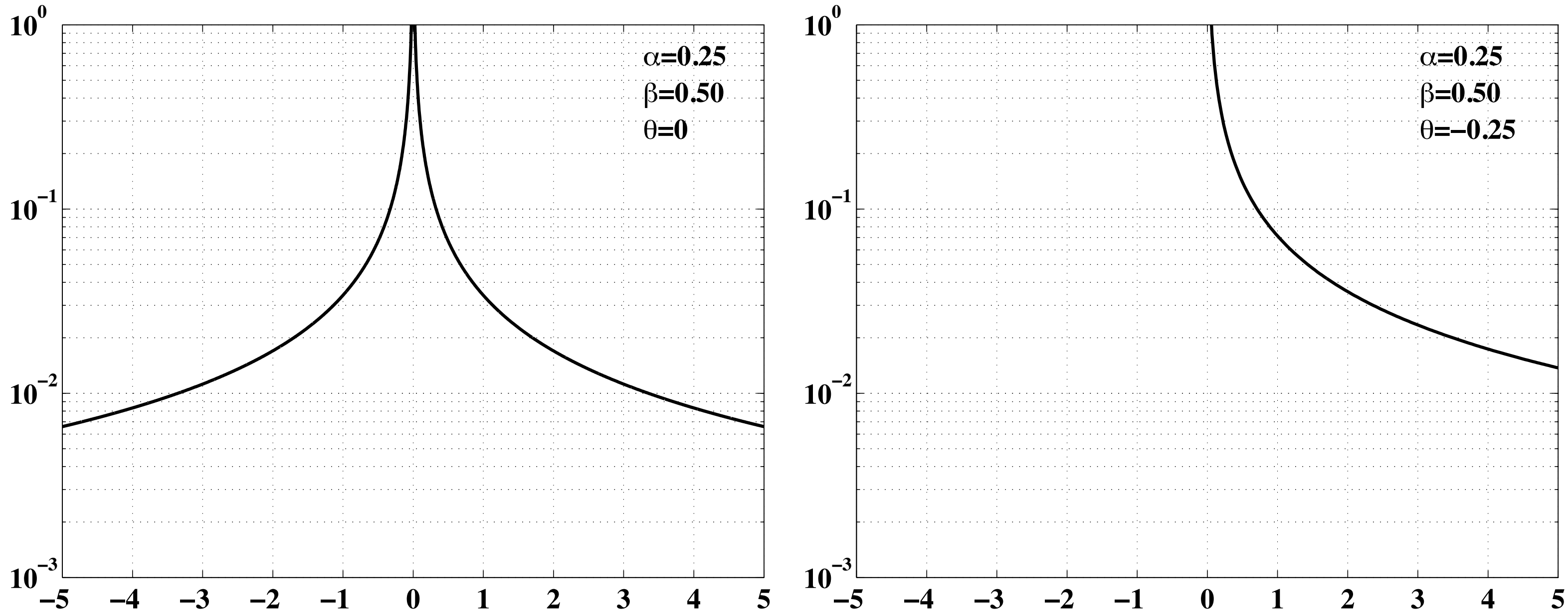
The Lévy probability density for $\alpha = 1$ and $\theta = 0$, $\theta = -0.99$



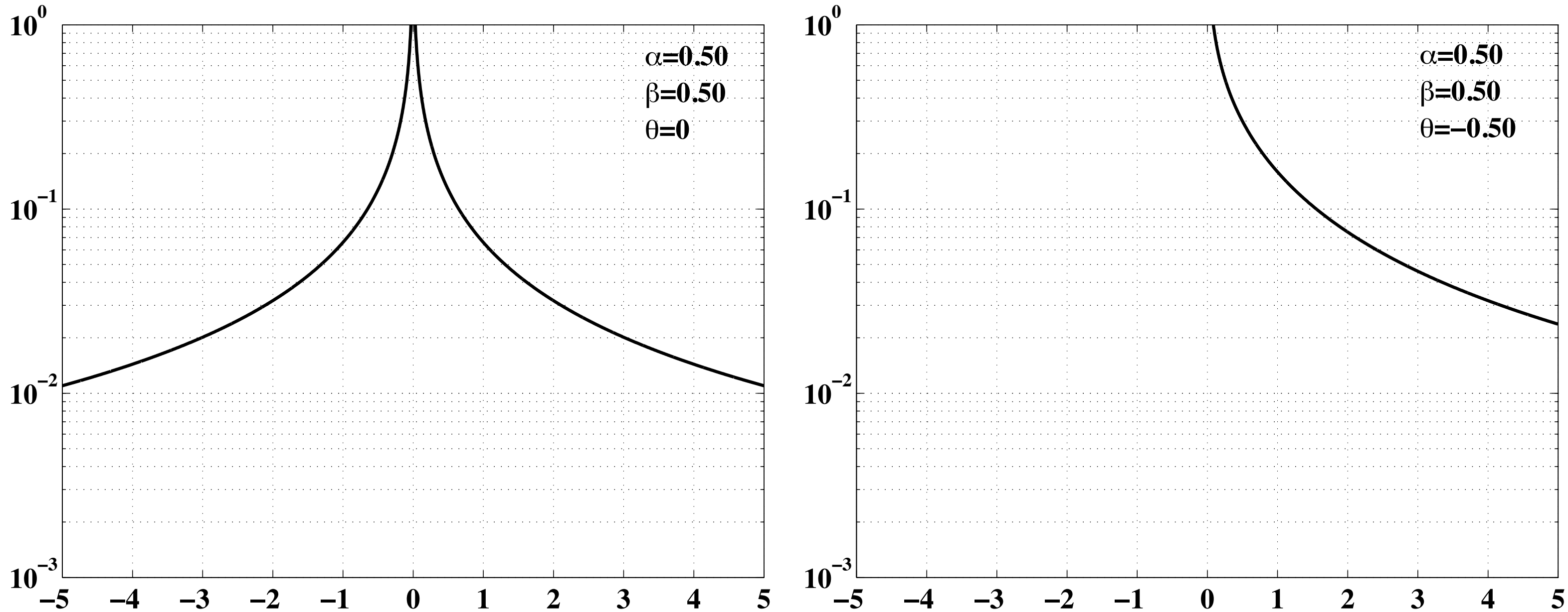
The Lévy probability density for $\alpha = 0.5$ and $\theta = 0$, $\theta = -0.50$



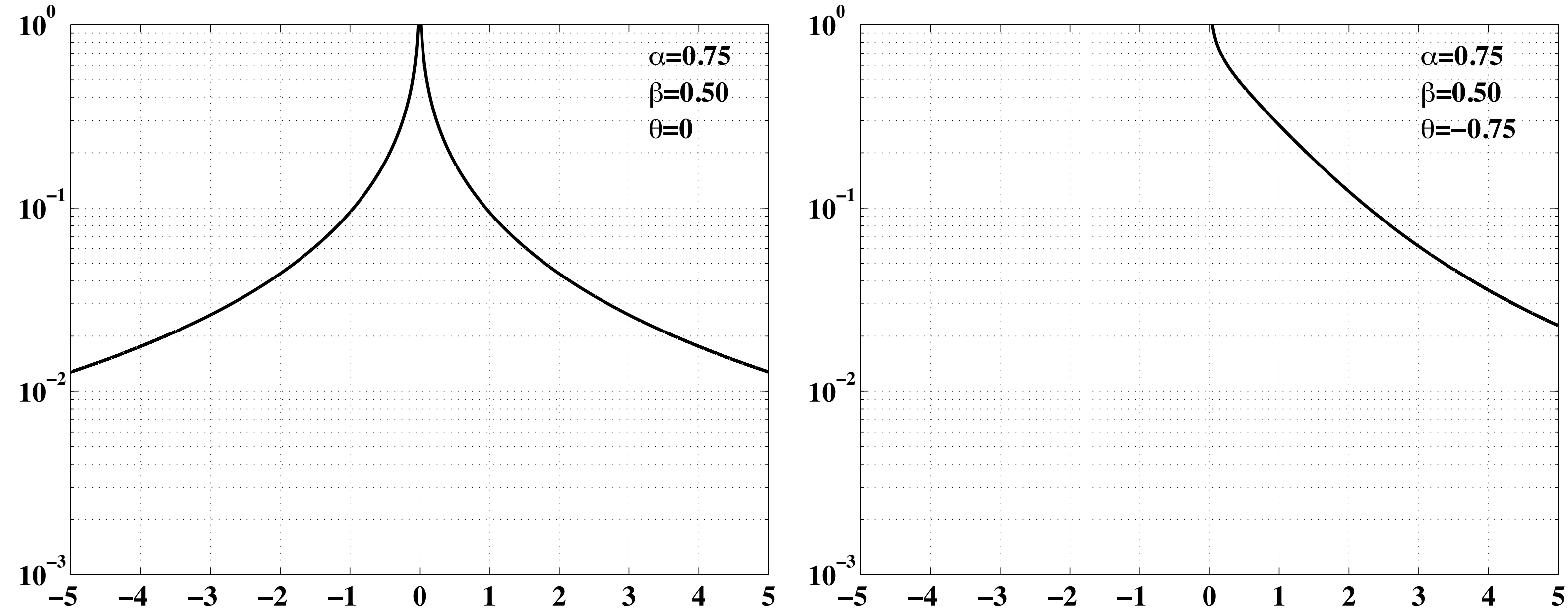
The M-Wright probability density for $\beta = 0.5$ **and** $\beta = 0.75$



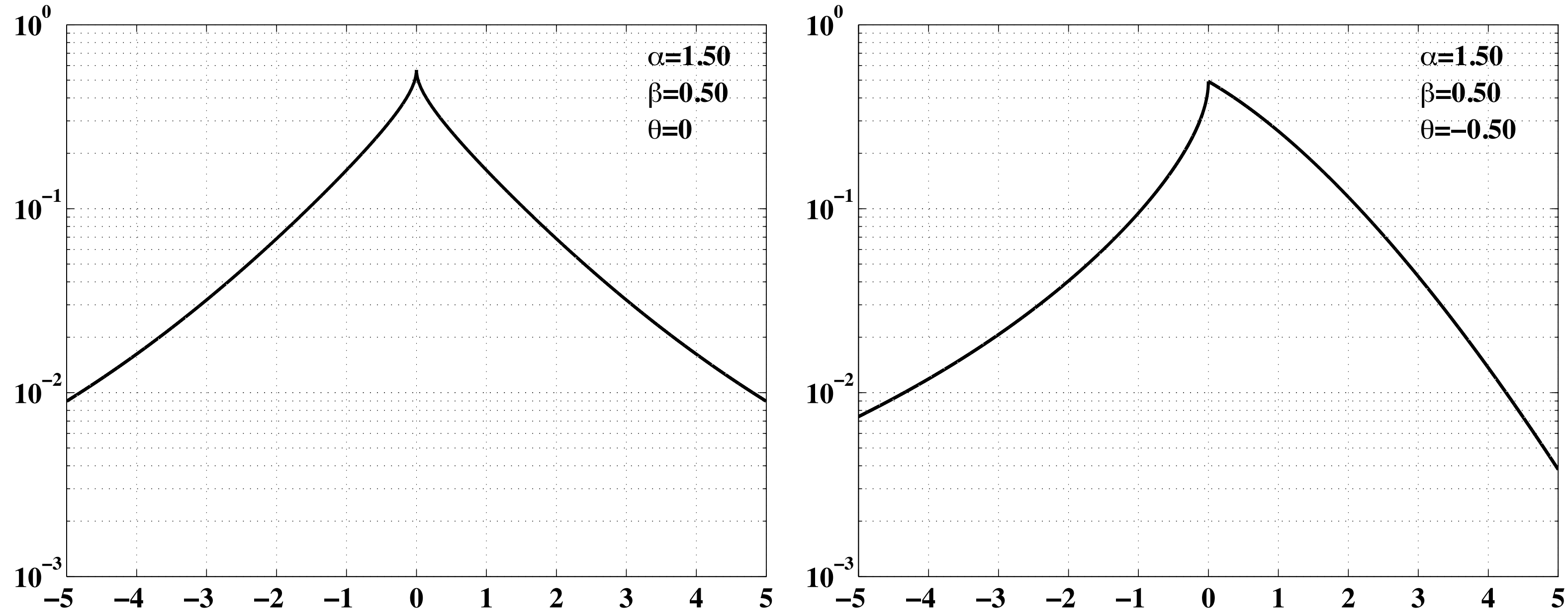
The probability density for $\alpha = 0.25, \beta = 0.5$ and $\theta = 0$, $\theta = -0.25$



The probability density for $\alpha = \beta = 0.5$ and $\theta = 0$, $\theta = -0.50$



The probability density for $\alpha = 0.75, \beta = 0.5$ and $\theta = 0$, $\theta = -0.75$



The probability density for $\alpha = 1.5, \beta = 0.5$ and $\theta = 0$, $\theta = -0.50$

The equations for time-fractional diffusion of distributed order

The R-L and the C forms in space-time domain

The **standard diffusion equation**, that in re-scaled non-dimensional variables reads

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)\,,\ x \in \mathbf{R}\,,\ t \in \mathbf{R}_0^+\,, \tag{2.1}$$

with $u(x,t)$ as the field variable, can be generalized by using the notion of fractional derivative of distributed order in time.

For this purpose, following the analysis for fractional relaxation of distributed order we need, to consider a function $p(\beta)$ that acts as weight for the order of differentiation $\beta \in (0,1]$ such that

$$p(\beta) \geq 0\,, \quad \text{and} \quad \int_0^1 p(\beta)\, d\beta = c > 0\,. \tag{2.2}$$

The positive constant c can be taken as 1

There are two possible forms of generalization depending if we use fractional derivatives intended in the R-L or C sense. Correspondingly we obtain the **time-fractional diffusion equation of distributed order** in the two forms:

$$\frac{\partial}{\partial t}\, u(x,t) = \int_0^1 p(\beta)\, {}_tD^{1-\beta} \left[\frac{\partial^2}{\partial x^2}\, u(x,t) \right] d\beta\,, \quad x \in \mathbf{R},\; t \geq 0\,, \quad (2.3a)$$

and

$$\int_0^1 p(\beta)\, \left[{}_tD_*^{\beta}\, u_*(x,t) \right] d\beta = \frac{\partial^2}{\partial x^2}\, u_*(x,t)\,, \quad x \in \mathbf{R},\; t \geq 0\,. \quad (2.3b)$$

From now on we shall restrict our attention on the fundamental solutions of Eqs. (2.3a)-(2.3b) so we understand that these equations are subjected to the initial condition $u(x,0^+) = u_*(x,0^+) = \delta(x)$. Since for distributed order the solution depends on the selected form (as we shall show hereafter), we now distinguish the two fractional equations and their fundamental solutions by decorating in the Caputo case the variable $u(x,t)$ with subscript $*$ as it is customary for the notation of the corresponding derivative.

Diffusion equations of distributed order of both types have been recently discussed by several authors. In some papers the authors have referred to the C and R-L forms as to **normal** and **modified** forms of the time-fractional diffusion equation of distributed order, respectively.

For a thorough general study of fractional pseudo-differential equations of distributed order let us cite the paper by Umarov and Gorenflo (2005). For a relationship with the Continuous Random Walk models we may refer to the paper by Gorenflo and Mainardi (2005).

The RL and C forms in Fourier-Laplace domain

The fundamental solutions for the time-fractional diffusion equations (2.3a)-(2.3b) can be obtained by applying in sequence the Fourier and Laplace transforms to them. We write, for generic functions $v(x)$ and $w(t)$, these transforms as follows:

$$\mathcal{F}\{v(x);\kappa\} = \widehat{v}(\kappa) := \int_{-\infty}^{+\infty} \mathrm{e}^{\,i\kappa x}\, v(x)\, dx\,,\ \ \kappa \in \mathbf{R}\,,$$

$$\mathcal{L}\{w(t);s\} = \widetilde{w}(s) := \int_{0}^{+\infty} \mathrm{e}^{\,-st}\, w(t)\, dt\,,\ \ s \in \mathbf{C}\,.$$

Then, in the Fourier-Laplace domain our Cauchy problems [with $u(x,0^+) = u_*(x,0^+) = \delta(x)$], after applying formulas for the Laplace transform appropriate to the R-L and C fractional derivatives, see (A.8') and (A.9), and observing $\widehat{\delta}(\kappa) \equiv 1$ appear in the two forms

$$s\widehat{\widetilde{u}}(\kappa,s) - 1 = -\kappa^2 \left[\int_0^\infty p(\beta) s^{1-\beta}\, d\beta\right] \widehat{\widetilde{u}}(\kappa,s)\,, \tag{2.4a}$$

$$\left[\int_0^\infty p(\beta) s^{\beta}\, d\beta\right] \widehat{\widetilde{u}}(\kappa,s) - \int_0^\infty p(\beta) s^{\beta-1}\, d\beta = -\kappa^2\, \widehat{\widetilde{u}}(\kappa,s)\,. \tag{2.4b}$$

Then, introducing the relevant functions

$$A(s) = \int_0^1 p(\beta)\, s^{1-\beta}\, d\beta\,, \tag{2.5a}$$

$$B(s) = \int_0^1 p(\beta)\, s^{\beta}\, d\beta\,, \tag{2.5b}$$

we then get for the R-L and C cases the Fourier-Laplace representation of the corresponding fundamental solutions:

$$\widehat{\widetilde{u}}(\kappa,s) = \frac{1}{s+\kappa^2 A(s)} = \frac{1/A(s)}{\kappa^2 + s/A(s)}\,, \tag{2.6a}$$

$$\widehat{\widetilde{u}}_*(\kappa,s) = \frac{B(s)/s}{\kappa^2 + B(s)}\,. \tag{2.6b}$$

From Eqs. (2.6a)-(2.6b) we recognize that the passage between the R-L and the C form can be carried out by the transformation

$$\{\mathrm{C}: \ B(s)\} \Longleftrightarrow \left\{\text{R-L}: \ \frac{s}{A(s)}\right\} . \tag{2.7}$$

We note that in the particular case of time fractional diffusion of single order β_0 ($0 < \beta_0 \leq 1$) we have $p(\beta) = \delta(\beta - \beta_0)$ hence in (2.5a): $A(s) = s^{1-\beta_0}$, in (2.5b): $B(s) = s^{\beta_0}$, so that $B(s) \equiv s/A(s)$. Then, Eqs. (2.6a) and (2.6b) provide the same result

$$\widehat{\widetilde{u}}(\kappa, s) \equiv \widehat{\widetilde{u}}_*(\kappa, s) = \frac{s^{\beta_0 - 1}}{\kappa^2 + s_0^{\beta}} . \tag{2.8}$$

This is consistent with the well-known result according to which the two forms are equivalent for the single order case. However, for a generic order distribution, the Fourier-Laplace representations (2.6a) (2.6b) are different so they produce in the space-time domain different fundamental solutions, that however are interrelated in some way in view of the transformation (2.7).

The variance of the fundamental solutions

General considerations

Before trying to get the fundamental solutions in the space-time domain to be obtained by a double inversion of the Fourier-Laplace transforms, it is worth to outline the expressions of their second moment (the variance) since these can be derived from Eqs. (2.6a)-(2.6b) by a single Laplace inversion, as it is shown hereafter. We recall that the time evolution of the variance is relevant for classifying the type of diffusion.

Denoting for the two forms

$$\text{R-L} \,:\; \sigma^2(t) := \int_{-\infty}^{+\infty} x^2\, u(x,t)\, dx\,,\; \text{C} \,:\; \sigma_*^2(t) := \int_{-\infty}^{+\infty} x^2\, u_*(x,t)\, dx\,, \tag{3.1}$$

we easily recognize that

$$\text{R-L} \,:\; \sigma^2(t) = -\frac{\partial^2}{\partial\kappa^2}\, \widehat{u}(\kappa=0,t)\,,\; \text{C} \,:\; \sigma_*^2(t) = -\frac{\partial^2}{\partial\kappa^2}\, \widehat{u}_*(\kappa=0,t)\,. \tag{3.2}$$

As a consequence we need to invert only Laplace transforms taking into account the behaviour of the Fourier transform for κ near zero.
For <u>the R-L case</u> we get from Eq. (2.6a),

$$\widehat{\widetilde{u}}(\kappa, s) = \frac{1}{s} \left(1 - \kappa^2 \frac{A(s)}{s} + \dots\right) ,$$

so we obtain

$$\widetilde{\sigma^2}(s) = -\frac{\partial^2}{\partial\kappa^2} \widehat{\widetilde{u}}(\kappa = 0, s) = \frac{2A(s)}{s^2} . \qquad (3.3a)$$

For <u>the C-case</u> we get from Eq. (2.6b)

$$\widehat{\widetilde{u}}_*(\kappa, s) = \frac{1}{s} \left(1 - \kappa^2 \frac{1}{B(s)} + \dots\right) ,$$

so we obtain

$$\widetilde{\sigma^2}_*(s) = -\frac{\partial^2}{\partial\kappa^2} \widehat{\widetilde{u_*}}(\kappa = 0, s) = \frac{2}{s\, B(s)} . \qquad (3.3b)$$

Except for the single order diffusion, were we recover the well-know result

$$\sigma^2(t) \equiv \sigma_*^2(t) = 2\,\frac{t^{\,\beta_0}}{\Gamma(\beta_0+1)}\,, \quad 0 < \beta_0 \leq 1\,, \tag{3.4}$$

for a generic order distribution, we expect that the time evolution of the variance substantially depends on the chosen (R-L or C) form.

We shall now concentrate our interest to some typical choices for the weight function $p(\beta)$ that characterizes the time-fractional diffusion equations of distributed order (2.3a) and (2.3b). This will allow us to compare the results for the R-L form and for the C form.

Fractional diffusion of double-order

First, we consider the choice

$$p(\beta) = p_1\delta(\beta - \beta_1) + p_2\delta(\beta - \beta_2)\,, \quad 0 < \beta_1 < \beta_2 \le 1\,, \tag{3.5}$$

where the constants p_1 and p_2 are both positive, conveniently restricted to the normalization condition $p_1 + p_2 = 1$.

Then for <u>the R-L case</u> we have

$$A(s) = p_1\, s^{1-\beta_1} + p_2\, s^{1-\beta_2}\,, \tag{3.6a}$$

so, in virtue of (3.3a), we have

$$\widetilde{\sigma^2}(s) = 2\,p_1\, s^{-(1+\beta_1)} + 2\,p_2\, s^{-(1+\beta_2)}\,. \tag{3.7a}$$

Finally the Laplace inversion yields, see and compare [**?**, **?**],

$$\sigma^2(t) = 2\,p_1\,\frac{t^{\beta_1}}{\Gamma(\beta_1+1)} + 2\,p_2\,\frac{t^{\beta_2}}{\Gamma(\beta_2+1)} \sim \begin{cases} 2p_1\,\dfrac{t^{\beta_1}}{\Gamma(1+\beta_1)}\,, & t \to 0^+\,, \\ 2p_2\,\dfrac{t^{\beta_2}}{\Gamma(1+\beta_2)}\,, & t \to +\infty\,. \end{cases} \tag{3.8a}$$

Similarly, for <u>the C case</u> we have

$$B(s) = p_1 \, s^{\beta_1} + p_2 \, s^{\beta_2} \,, \tag{3.6b}$$

so, in virtue of (3.3b),

$$\widetilde{\sigma^2}_*(s) = \frac{2}{p_1 \, s^{(1+\beta_1)} + p_2 \, s^{(1+\beta_2)}} \,. \tag{3.7b}$$

Finally the Laplace inversion yields

$$\sigma^2_*(t) = \frac{2}{p_2} \, t^{\beta_2} \, E_{\beta_2-\beta_1,\beta_2+1}\left(-\frac{p_1}{p_2} \, t^{\beta_2-\beta_1}\right) \sim \begin{cases} \dfrac{2}{p_2} \, \dfrac{t^{\beta_2}}{\Gamma(1+\beta_2)} \,, & t \to 0^+ \,, \\ \dfrac{2}{p_1} \, \dfrac{t^{\beta_1}}{\Gamma(1+\beta_1)} \,, & t \to +\infty \,. \end{cases} \tag{3.8b}$$

Then we see that for the R-L case we have an explicit combination of two power laws: the smallest exponent(β_1) dominates at small times whereas the largest exponent (β_2) dominates at large times. For the C case we have a Mittag-Leffler function in two parameters so we have a combination of two power laws only asymptotically for small and large times; precisely we get a behaviour opposite to the previous one, so the largest exponent(β_2) dominates at small times whereas the smallest exponent (β_1) dominates at large times. We can derive the above asymptotic behaviours directly from the Laplace transforms (3.7a)-(3.7b) by applying the Tauberian theory for Laplace transforms.

In fact for the R-L case we note that for $A(s)$ in (3.6a) $s^{1-\beta_1}$ is negligibly small in comparison with $s^{1-\beta_2}$ for $s \to 0^+$ and, viceversa, $s^{1-\beta_2}$ is negligibly small in comparison to $s^{1-\beta_1}$ for $s \to +\infty$.

Similarly for the C case we note that for $B(s)$ in (3.6b) s^{β_2} is negligibly small in comparison to s^{β_1} for $s \to 0^+$ and, viceversa, s^{β_1} is negligibly small in comparison s^{β_2} for $s \to +\infty$.

Fractional diffusion of uniformly distributed order

We consider the choice

$$p(\beta) = 1\,, \quad 0 < \beta < 1\,. \tag{3.9}$$

For <u>the R-L case</u> we have

$$A(s) = s\int_0^1 s^{-\beta}\,d\beta = \frac{s-1}{\log s}\,, \tag{3.10a}$$

hence, in virtue of (3.3a),

$$\widetilde{\sigma^2}(s) = 2\left[\frac{1}{s\log s} - \frac{1}{s^2\log s}\right]\,. \tag{3.11a}$$

$$\sigma^2(t) = 2\,[\nu(t,0) - \nu(t,1)] \sim \begin{cases} 2/\log\,(1/t)\,, & t \to 0\,, \\ 2t/\log t\,, & t \to \infty\,, \end{cases} \tag{3.12a}$$

$$\nu(t,a) := \int_0^\infty \frac{t^{a+\tau}}{\Gamma(a+\tau+1)}\,d\tau\,, \quad a > -1\,.$$

For the C case we have

$$B(s) = \int_0^1 s^\beta \, d\beta = \frac{s-1}{\log s}\,, \tag{3.10b}$$

hence, in virtue of (3.3b),

$$\widetilde{\sigma_*^2}(s) = \frac{2}{s}\,\frac{\log s}{s-1}\,. \tag{3.11b}$$

Then, by inversion,

$$\sigma_*^2(t) = 2\left[\log t + \gamma + \mathrm{e}^{\,t}\,\mathcal{E}_1(t)\right] \sim \begin{cases} 2t\log(1/t), & t \to 0, \\ 2\log(t), & t \to \infty\,, \end{cases} \tag{3.12b}$$

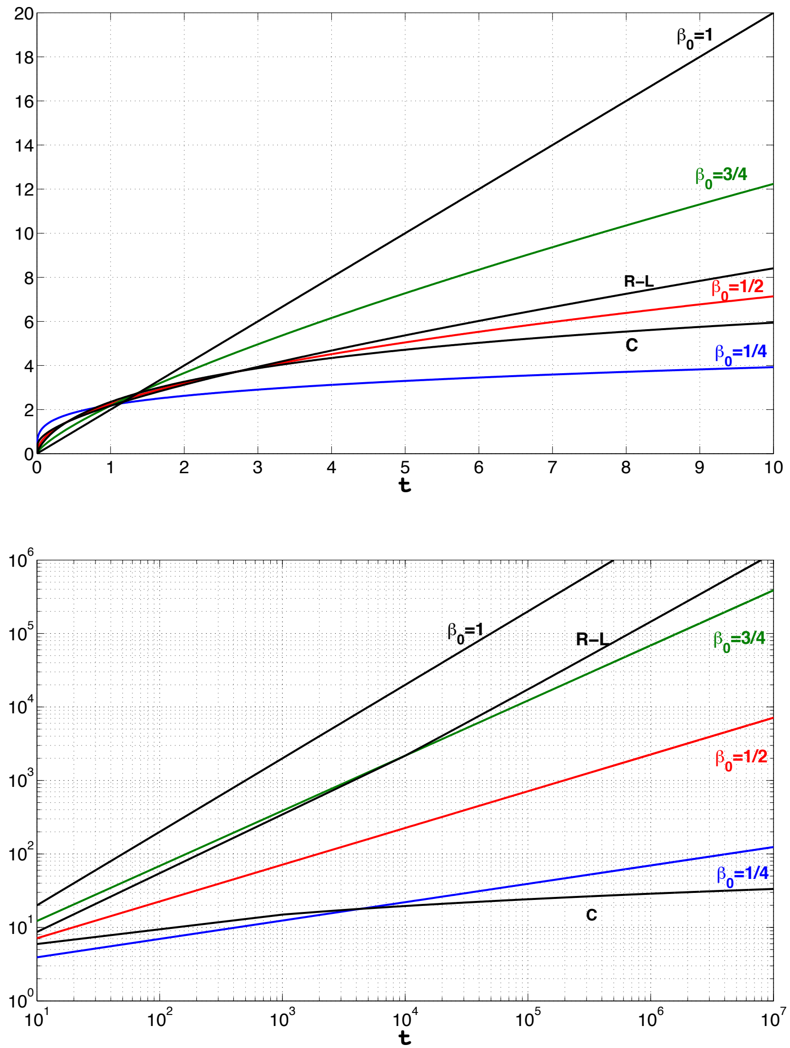
where

$$\mathcal{E}_1(t) := \int_t^\infty \frac{\mathrm{e}^{-u}}{u}\,du = \mathrm{e}^{-t}\int_0^\infty \frac{\mathrm{e}^{-u}}{u+t}\,du$$

denotes the exponential integral function and $\gamma = 0.57721...$ is the so-called Euler-Mascheroni constant.

For the uniform distribution we find it instructive to compare the time evolution of the variance for the R-L and C forms with that corresponding to a few of single orders.

In the following Figures we consider moderate times ($0 \le t \le 10$) using linear scales, and large times ($10^1 \le t \le 10^7$) using logarithmic scales.



Variance versus t for the uniform order distribution in R-L and C forms compared with some single order case. Top: $0 \leq t \leq 10$ (linear scales); Bottom: $10^1 \leq t \leq 10^7$ (logarithmic scales).

Evaluation of the fundamental solutions

The two strategies

In order to determine the fundamental solutions $u(x,t)$ and $u_*(x,t)$ in the space-time domain we can follow two alternative strategies related to the order in carrying out the Fourier-Laplace in (2.6a) and (2.6b)
(S1) : invert the Fourier transforms getting $\widetilde{u}(x,s)$, $\widetilde{u}_*(x,s)$ and then invert the remaining Laplace transforms;
(S2) : invert the Laplace transform getting $\widehat{u}(\kappa,t)$, $\widehat{u}_*(\kappa,t)$ and then invert the remaining Fourier transform.

Before considering the general case of time-fractional diffusion of distributed order, we prefer to briefly recall the determination of the fundamental solution $u(x,t)$ (common for both the R-L and C forms) for the single order case.

The single order diffusion

For the time-fractional diffusion equation of single order β_0 the strategy (S1) yields the Laplace transform

$$\widetilde{u}(x,s) = \frac{s^{\beta_0/2-1}}{2}\, \mathrm{e}^{-|x|s^{\beta_0/2}}\,, \quad 0 < \beta_0 \le 1\,. \tag{4.1}$$

Such strategy was adopted by Mainardi (1993)-(1997) to obtain the Green function in the form

$$u(x,t) = t^{-\beta_0/2}\, U\left(|x|/t^{\beta_0/2}\right)\,, \quad -\infty < x < +\infty\,, \quad t \ge 0\,, \tag{4.2}$$

where the variable $X := x/t^{\beta_0/2}$ acts as **similarity variable** and the function $U(x) := u(x,1)$ denotes the **reduced Green function**.

Restricting from now on our attention to $x \geq 0$, the solution turns out

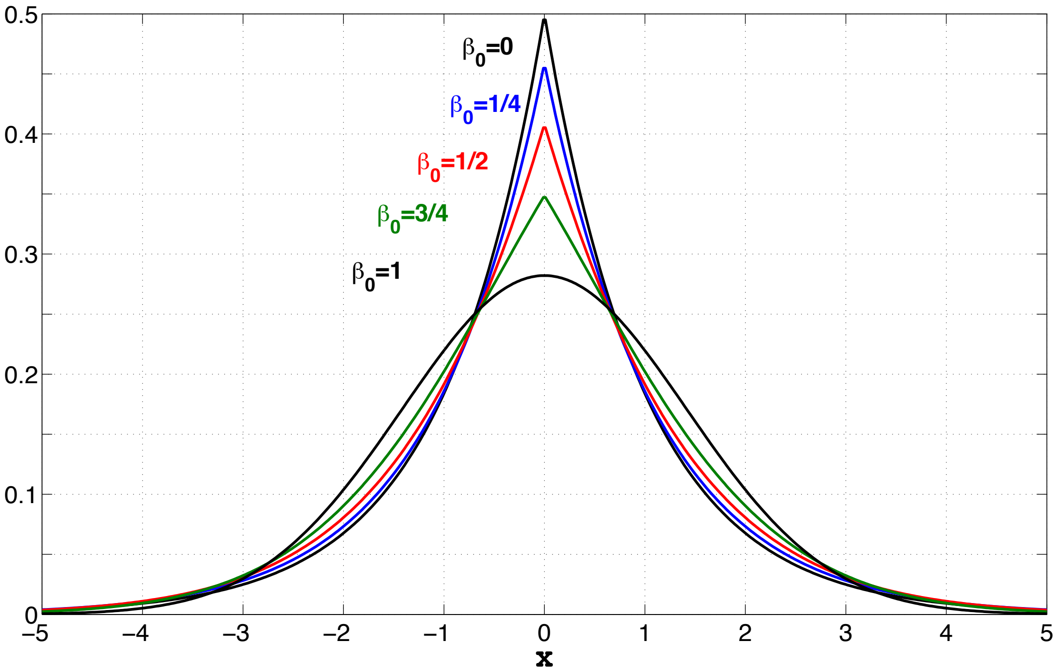
$$\begin{aligned} U(x) &= \frac{1}{2}\, M_{\beta_0/2}(x) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-x)^k}{k!\,\Gamma[-\beta_0 k/2 + (1-\beta_0/2)]} \\ &= \frac{1}{2\pi}\sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\,\Gamma[(\beta_0(k+1)/2]\,\sin[(\pi\beta_0(k+1)/2]\,, \end{aligned} \tag{4.3}$$

where $M_{\beta_0/2}(x)$ is an an entire transcendental function of the Wright type. Since the fundamental solution has the peculiar property to be self-similar it is sufficient to consider the reduced Green function $U(x)$. In the Figure we show the graphical representations of $U(x)$ for different orders ranging from $\beta_0 = 0$, for which we recover the Laplace density

$$U(x) = \frac{1}{2}\,\mathrm{e}^{-|x|}\,, \tag{4.4}$$

to $\beta_0 = 1$, for which we recover the Gaussian density (of variance $\sigma^2 = 2$)

$$U(x) = \frac{1}{2\sqrt{\pi}}\,\mathrm{e}^{-x^2/4}\,. \tag{4.5}$$



The reduced Green function $U(x) = \frac{1}{2} M_{\beta_0/2}(x))$ versus x (in the interval $|x| \leq 5$), for $\beta_0 = 0, 1/4, 1/2, 3/4, 1$.

The Strategy (S2): yields the Fourier transform.

$$\widehat{u}(\kappa,t) = E_{\beta_0}\left(-\kappa^2 t^{\beta_0}\right)\,, \quad 0 < \beta_0 \le 1\,, \tag{4.6}$$

where E_{β_0} denotes the Mittag-Leffler function. The strategy (S2) has been followed by Gorenflo, Iskenderov and Luchko (2000)and by Mainardi, Luchlo and Pagnini (2001) to obtain the Green functions of the more general space-time-fractional diffusion equations (of single order), and requires to invert the Fourier transform by using the machinery of the Mellin convolution and the Mellin-Barnes integrals. Restricting ourselves here to recall the final results, the reduced Green function for the time-fractional diffusion equation now appears, for $x \ge 0$, in the form:

$$\begin{aligned} U(x) &= \frac{1}{\pi}\int_0^\infty \cos(\kappa x)\, E_{\beta_0}\left(-\kappa^2\right)\, d\kappa \\ &= \frac{1}{2x}\,\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\Gamma(1-s)}{\Gamma(1-\beta_0 s/2)}\, x^{\,s}\, ds\,, \end{aligned} \tag{4.7}$$

with $0 < \sigma < 1$.

By solving the Mellin-Barnes integrals using the residue theorem, we arrive at the same power series (4.3) of the M-Wright function. Both strategies allow us to prove that the Green function is non-negative and normalized, so it can be interpreted as a spatial probability density evolving in time with the similarity law (4.2).

For readers' convenience we like to mention other papers dealing with the fundamental solutions of fractional diffusion equations (of single ordrer); a non exhaustive list includes AnhLeonenko (2001), Eidelman-Kochubei (2004), Hanyga (2002), Hilfer (2000), Kochubei (1990), Metzler (1994), Metzler-Klafter (2000),Saichev-Zaslavsky (1997), Schneider-Wyss (1989), Zaslavsky (2002) and references therein.

The distributed order diffusion

Similarly with the single order diffusion, also for the cases of distributed order we can follow either strategy (S1) or strategy (S2). Here we follow the strategy (S1). This choice implies to recall the Fourier transform pair (a straightforward exercise in complex analysis based on residue theorem and Jordan's lemma)

$$\frac{c}{d+\kappa^2} \stackrel{\mathcal{F}}{\leftrightarrow} \frac{c}{2d^{1/2}}\,\mathrm{e}^{-|x|d^{1/2}}\,, \quad d>0\,. \tag{4.8}$$

In fact we recognize by comparing (4.8) with (2.6a)-(2.6b) that for the RL and C forms we have

$$\text{R-L}: \begin{cases} c = c(s) := 1/A(s) \\ d = d(s) := s/A(s) \end{cases} \qquad \text{C}: \begin{cases} c = c(s) := B(s)/s\,, \\ d = d(s) := B(s) \end{cases} \tag{4.9}$$

Now we have to invert the Laplace transforms obtained inserting (4.9) in the R.H.S of (4.8).

For <u>the R-L case</u> we have:

$$\widetilde{u}(x,s) = \frac{1}{2[sA(s)]^{1/2}} \exp\left\{-|x|[s/A(s)]^{1/2}\right\} . \qquad (4.10a)$$

For <u>the C case</u> we have:

$$\widetilde{u}_*(x,s) = \frac{[B(s)]^{1/2}}{2s} \exp\left\{-|x|[B(s)]^{1/2}\right\} . \qquad (4.10b)$$

Following a standard procedure in complex analysis, the Laplace inversion requires the integration along the borders of the negative real semi-axis in the s-complex cut plain; in fact this semi-axis, defined by $s = r\mathrm{e}^{\pm i\pi}$ with $r > 0$ turns out the branch-cut common for the functions $s^{1-\beta}$ (present in $A(s)$for the RL form) and s^{β} (present in $B(s)$ for the C form). Then, in virtue of the Titchmarsh theorem on Laplace inversion, we get the representations in terms of real integrals of Laplace type.

For <u>the R-L case</u> we get

$$u(x,t) = -\frac{1}{\pi} \int_0^\infty \mathrm{e}^{-rt}\, \mathrm{Im}\left\{\widetilde{u}\left(x, r\mathrm{e}^{i\pi}\right)\right\}\, dr\,, \qquad (4.11a)$$

where, in virtue of (4.10a), we must know $A(s)$ along the ray $s = r\,\mathrm{e}^{i\pi}$ with $r > 0$. We write

$$A\left(r\,\mathrm{e}^{i\pi}\right) = \rho\,\cos(\pi\gamma) + i\rho\sin(\pi\gamma)\,, \qquad (4.12a)$$

where

$$\begin{cases} \rho = \rho(r) = \left|A\left(r\,\mathrm{e}^{i\pi}\right)\right|, \\ \gamma = \gamma(r) = \dfrac{1}{\pi}\arg\left[A\left(r\,\mathrm{e}^{i\pi}\right)\right]. \end{cases} \qquad (4.13a)$$

Similarly for the <u>C case</u> we obtain

$$u_*(x,t) = -\frac{1}{\pi}\int_0^{\infty} \mathrm{e}^{-rt}\, \mathrm{Im}\left\{\widetilde{u}_*\left(x, r\mathrm{e}^{i\pi}\right)\right\}\, dr\,, \tag{4.11b}$$

where, in virtue of (4.10b), we must know $B(s)$ along the ray $s = r\,\mathrm{e}^{i\pi}$ with $r > 0$. We write

$$B\left(r\,\mathrm{e}^{\,i\pi}\right) = \rho_*\, \cos(\pi\gamma_*) + i\rho_*\sin(\pi\gamma_*)\,, \tag{4.12b}$$

where

$$\begin{cases} \rho_* = \rho_*(r) = \left|B\left(r\,\mathrm{e}^{i\pi}\right)\right|\,, \\ \gamma_* = \gamma_*(r) = \dfrac{1}{\pi}\arg\left[B\left(r\,\mathrm{e}^{i\pi}\right)\right]\,. \end{cases} \tag{4.13b}$$

As a consequence we formally write the required fundamental solutions as

$$u(x,t) = \int_0^\infty \mathrm{e}^{-rt}\, P(x,r)\, dr\,, \;\; P(x,r) = -\frac{1}{\pi}\,\mathrm{Im}\left\{\widetilde{u}\left(x, r\mathrm{e}^{i\pi}\right)\right\}\,, \quad (4.14a)$$

and

$$u_*(x,t) = \int_0^\infty \mathrm{e}^{-rt}\, P_*(x,r)\, dr\,, \;\; P_*(x,r) = -\frac{1}{\pi}\,\mathrm{Im}\left\{\widetilde{u}_*\left(x, r\mathrm{e}^{i\pi}\right)\right\}\,, \quad (4.14b)$$

where the functions $P(x,r)$ and $P_*(x,r)$ must be derived by using Eqs. (4.10a)-(4.14a) and Eqs. (4.10b)-(4.14b), respectively. We recognize that, in view of the transformation (2.7), the expressions of P and P_* are related to each other by the transformation

$$\rho_*(r) \Longleftrightarrow r/\rho(r)\,, \quad \gamma_*(r) \Longleftrightarrow 1-\gamma(r)\,. \quad (4.15)$$

We then limit ourselves to provide the explicit expression for the C form

$$
\begin{aligned}
P_*(x,r) &= \frac{1}{2\pi r}\,\mathrm{Im}\left\{\rho_*^{1/2}\,\mathrm{e}^{\,i\pi\gamma_*/2}\exp\left[-\,\mathrm{e}^{\,i\pi\gamma_*/2}\rho_*^{1/2}x\right]\right\} \\
&= \frac{1}{2\pi r}\,\rho_*^{1/2}\,\mathrm{e}^{\,-\rho_*^{1/2}x\cos(\pi\gamma_*/2)}\,\sin\left[\pi\gamma_*/2-\rho_*^{1/2}x\sin(\pi\gamma_*/2)\right]\,.
\end{aligned}
\tag{4.16}
$$

For the R-L form the corresponding expression of $P(x;r)$ is obtained from (4.16) by applying the transformation (4.15).

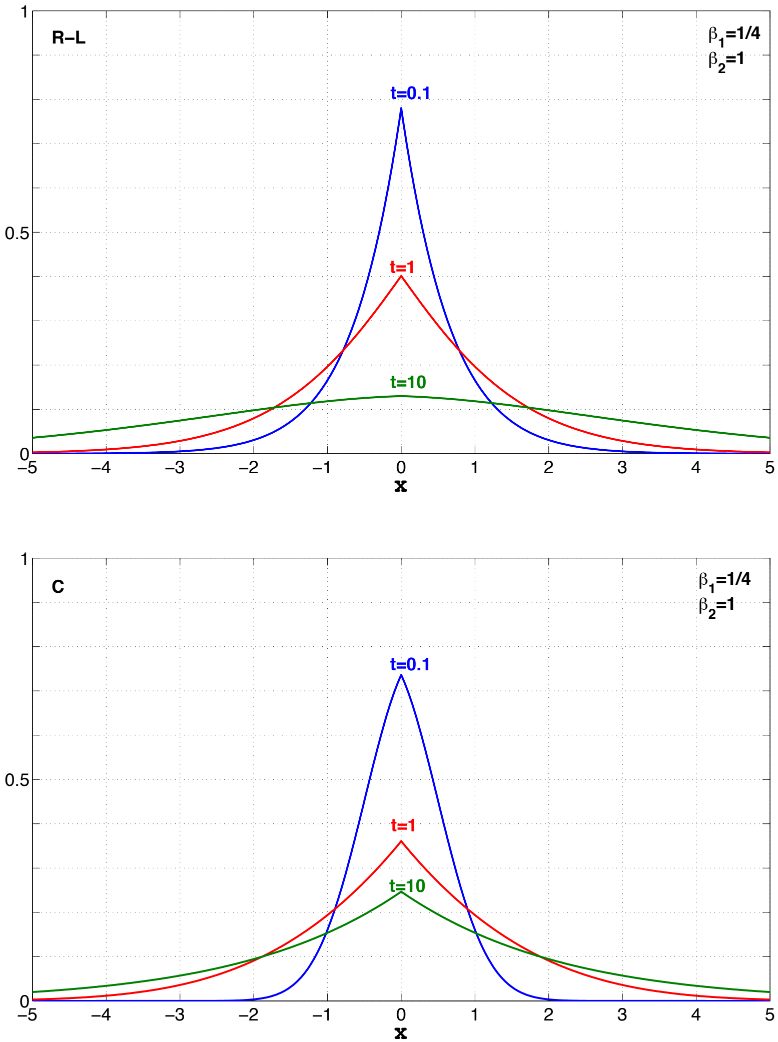
Herafter we exhibit some plots of the fundamental solutions for the two case studies considered in subsection 3.2 in order to point out the remarkable difference between the R-L and the C forms.

Plots of the fundamental solutions

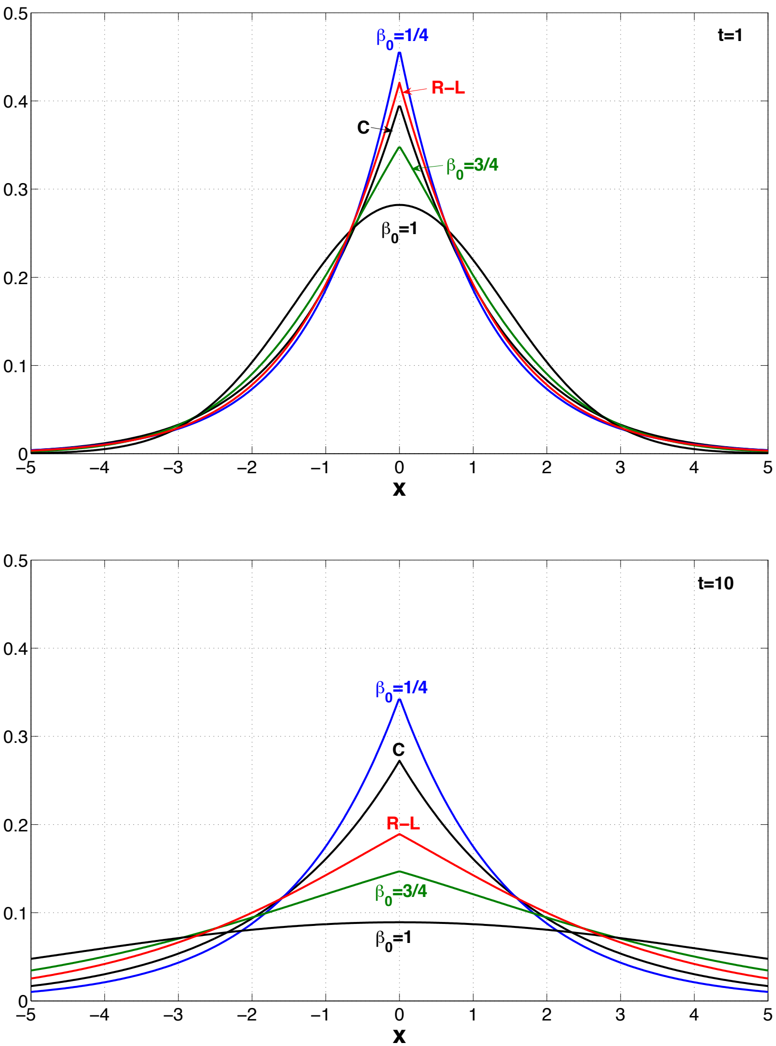
For the case of two orders, we chose $\{\beta_1 = 1/4, \beta_2 = 1\}$ in order to contrast the evolution of the fundamental solution for the R-L and the C forms.

In a following Figure we exhibit the plots of the corresponding solution versus x (in the interval $|x| \leq 5$), at different times, selected as $t = 0.1$, $t = 1$ and $t = 10$. In this limited spatial range we can note how the time evolution of the *pdf* depends on the different time-asymptotic behaviour of the variance, for the two forms, as stated in Eqs. (3.12a)-(3.12b), respectively.

For the uniform distribution, we find it instructive to compare in another Figure the solutions corresponding to R-L and C forms with the solutions of the fractional diffusion of a single order $\beta_0 = 1/4, 3/4, 1$ at fixed times, selected as $t = 1, 10$. We have skipped $\beta_0 = 1/2$ for a better view of the plots.



The fundamental solution versus x (in the interval $|x| \leq 5$), for the double-order distribution $\{\beta_1 = 1/4, \beta_2 = 1\}$ at times $t = 0.1, 1, 10$. Top: R-L form; Bottom: C form.



The fundamental solutions versus x (in the interval $|x| \leq 5$), for the uniform order distribution in R-L and C forms compared with the solutions for some cases of single order. Top: $t = 1$; Bottom: $t = 10$.

Conclusions on time-fractional diffusion of distributed order

We have investigated the time fractional diffusion equation with (discretely or continuously) distributed order between 0 and 1 in the Riemann-Liouville and in the Caputo forms, providing the Fourier-Laplace representation of the corresponding fundamental solutions.

Except for the case of a single order, for which the two forms are equivalent with a self-similar fundamental solution, for a general order distribution the equivalence and the self-similarity are lost. In particular the asymptotic behaviour of the fundamental solution and its variance at small and large times strongly depends on the selected approach.

We have considered two simple but noteworthy case-studies of distributed order, namely the case of a superposition of two different orders β_1 and β_2 and the case of a uniform order distribution. In the first case one of the orders dominates the time-asymptotics near zero, the other near infinity, but β_1 and β_2 change their roles when switching from the R-L form to the C form of the time-fractional diffusion.

The asymptotics for uniform order density is remarkably different, the extreme orders now being (roughly speaking) 0 and 1. We now meet super-slow and slightly super-fast time behaviours of the variance near zero and near infinity, again with the interchange of behaviours between the R-L and C form. We have clearly pointed out the above effects with the figures in sub-section 3.3, in particular the extremely slow growth of the variance as $t \to \infty$ for the C form.

After the analysis of the variance, that in practice requires only the inversion of a Laplace transform, we have considered the task of the double inversion of the Laplace-Fourier representation. For a general order distribution we were able to express the fundamental solution in terms of a Laplace integral in time with a kernel which depends on space and order distribution in a simple form, see Eqs. (4.14)-(4.16). For the two case studies the plots of the fundamental solutions (reported in sub-section 4.4) have shown their dependence on the different asymptotic behavior of the corresponding variance.