



Turkish National Meeting on Automatic Control  
(TOK 2013) , Sept. 25, 2013, Malatya, Turkey



# A Tutorial on Fractional Order Motion Control

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**MESA Lab** <http://mechatronics.ucmerced.edu>



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Part I: Overview of the Book and  
Fundamentals of Fractional Order Controls

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- Fractional Order Motion Controls

John Wiley & Sons, Inc.

Hardcover, 454 pages, December 2012

- Dr. Ying Luo

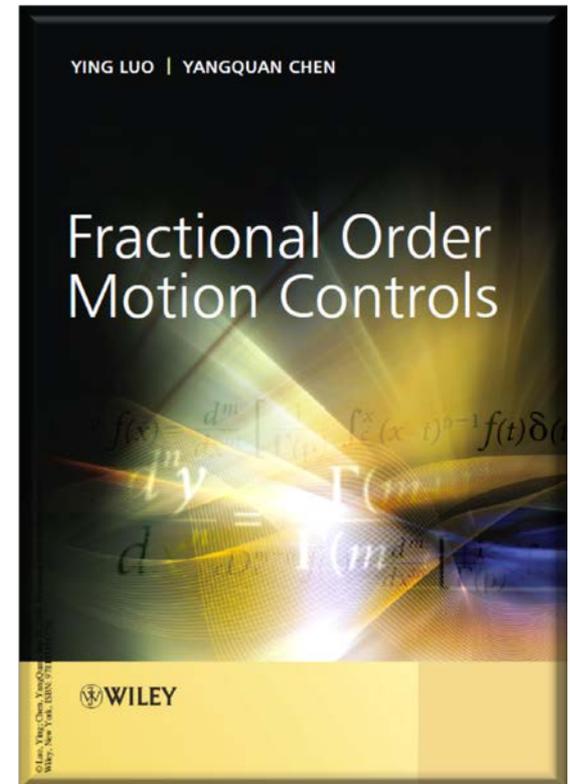
Automation Science and Engineering, South China  
University of Technology, Guangzhou, PR China

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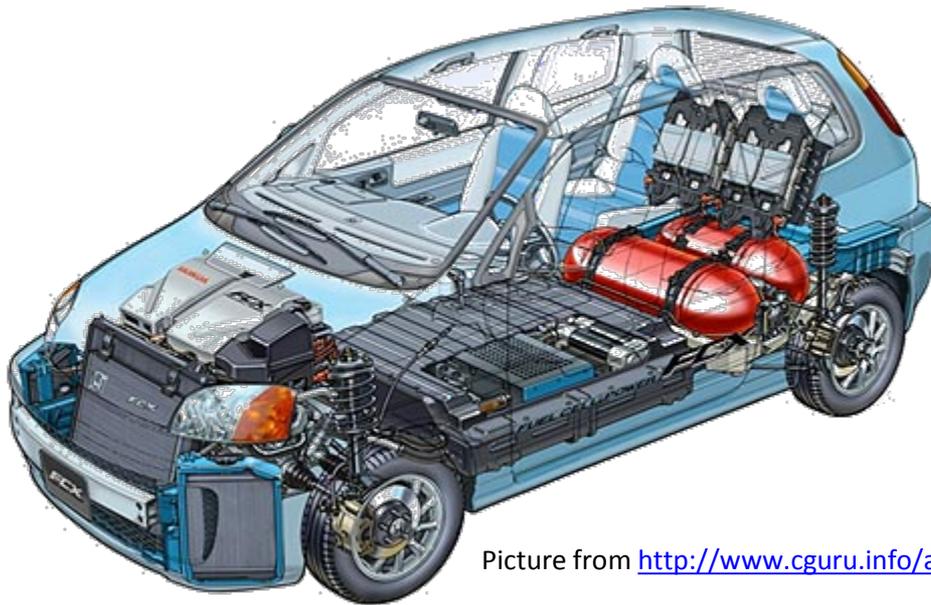
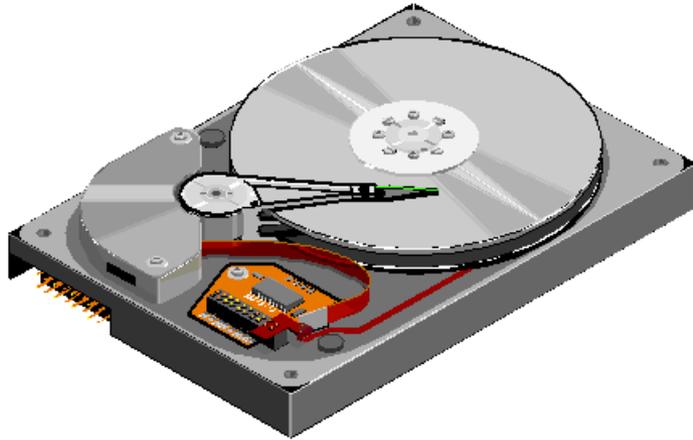
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## Contributions

- In the velocity and position motion systems, fractional calculus is applied to the FO modeling and FO controllers design in systematic ways to achieve better performances than the designed integer order controllers using traditional optimization methods. Stability and feasibility issues are analyzed and discussed for the fractional order controllers design.
- FO disturbance observer, FO adaptive feed-forward scheme, FO adaptive control and FO periodic adaptive learning control for external disturbance compensation are presented.
- This book documents the very first optimization approach of the proposed FO conditional integrator for nonlinear system controls. Meanwhile, some efforts of the fractional order control on nonlinearities, for example, friction and backlash, are presented.
- Simulation illustrations and/or experimental validations are demonstrated for all the proposed control schemes and approaches. The real applications for FO motion controls on unmanned aerial vehicles and hard-disk drives are presented in this book.



Picture from <http://www.cguru.info/automobile.htm>



Collection of resources for control tutorial

Control Tutorials for MATLAB and Simulink

<http://ctms.engin.umich.edu/CTMS/index.php?aux=Home>

Collection of resources for fractional calculus

Fractional Calculus Day @ UCMerced

<http://mechatronics.ucmerced.edu/node/68>

Applied Fractional Calculus (AFC) @ UCMerced

<http://mechatronics.ucmerced.edu/research/applied-fractional-calculus>

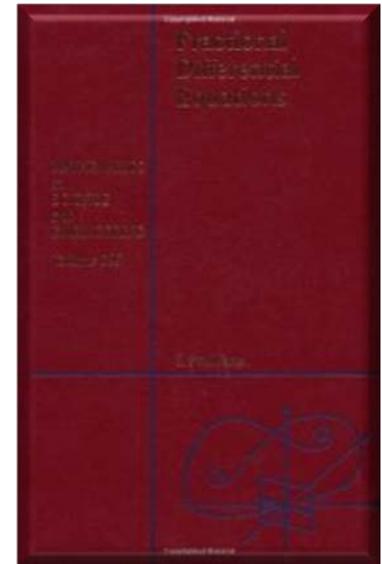
Applied Fractional Calculus (AFC) @ USU

<http://mechatronics.ece.usu.edu/foc/>

## Collection of resources for fractional calculus

Richard Magin, [Fractional Calculus in Bioengineering](#), ISBN: 978-1-56700-215-7, 684 pages, 2006.

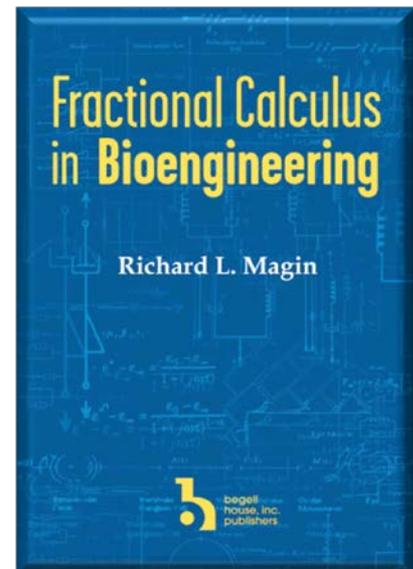
Igor Podlubny, [Fractional Differential Equations](#), 1st Edition, An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, 1998



Google scholar    [My Citations](#)

**Authors** 1-10 [Next >](#)

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	<p><b>Francesco Mainardi</b> Free Professor of Mathematical Physics, University of Bologna and INFN, I-40126 Bologna, ... Verified email at bo.infn.it Cited by 8346</p>		<p><b>Richard L. Magin</b> Professor of Bioengineering, University of Illinois at Chicago Verified email at uic.edu Cited by 6257</p>



- Crises in the foundations of mathematics push the development of math
- Paradox
  - What does  $n=1/2$  mean in the derivative of a function?

*Fractional Calculus* was born in 1695

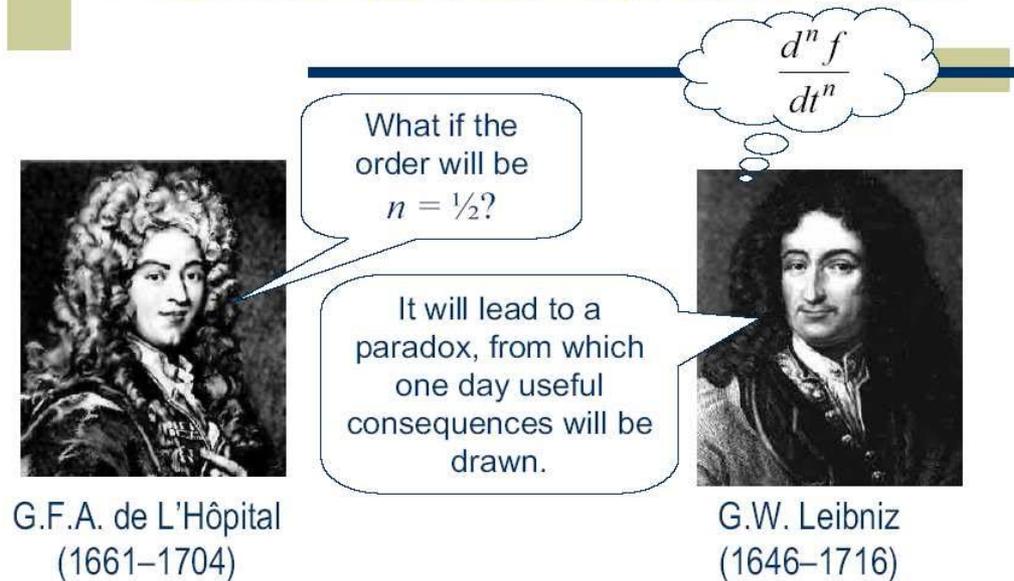


Figure: The born of fractional calculus

### Definition 1

Fractional Order **Cauchy** integral formula

$$D^\alpha f(t) = \frac{\Gamma(\alpha + 1)}{j2\pi} \int_C \frac{f(\tau)}{(\tau - t)^{\alpha+1}} d\tau, \quad (1)$$

where C is the closed-path that encircles the poles of the function f (t).

### Definition 2

**Grünwald-Letnikov** definition

$${}_a D_t^\alpha f(t) = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} f(t - jh), \quad (2)$$

where  $\binom{\alpha}{j}$  are the binomial coefficients; the subscripts to the left and right of D are the lower- and upper-bounds in the integral. The value  $\alpha$  can be positive or negative, corresponding to differentiation and integration, respectively.

## Definition 3

**Riemann-Liouville** definition

$${}_a D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad (3)$$

where  $0 < \alpha < 1$ , and  $a$  is the initial value. Let  $a = 0$ , the notation of integral can be simplified to  $D_t^{-\alpha} f(t)$ .

## Definition 4

**Caputo** definition

$${}_0 D_t^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{f^{(m+1)}(\tau)}{(t - \tau)^\alpha} d\tau, \quad (4)$$

where  $\alpha = m + \gamma$ ,  $m$  is an integer and  $0 < \gamma \leq 1$ .

## Fractional order differential equations

$$\begin{aligned} a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \dots + a_0 D^{\alpha_0} y(t) \\ = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \dots + b_0 D^{\beta_0} u(t), \end{aligned} \quad (5)$$

## Laplace transform of a fractional order derivative operator

$$D^\alpha \rightarrow s^\alpha \quad (6)$$

Where  $-1 \leq \alpha \leq n$ .

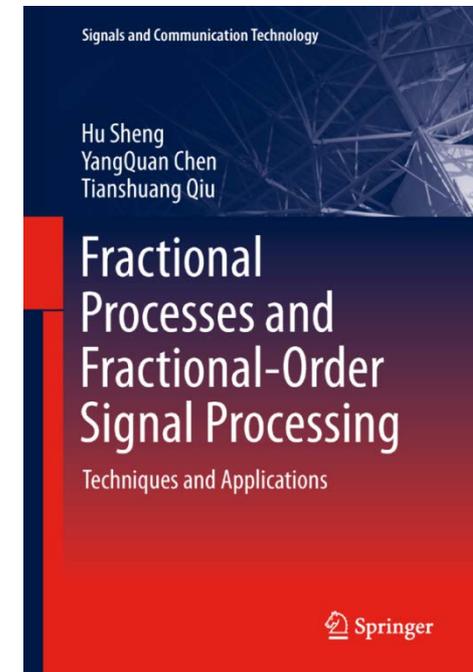
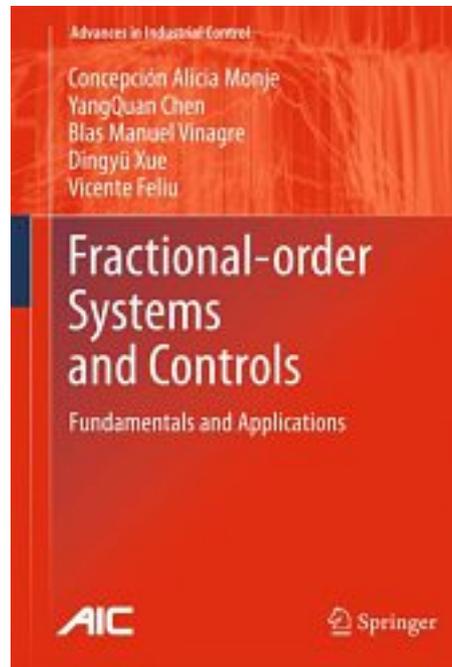
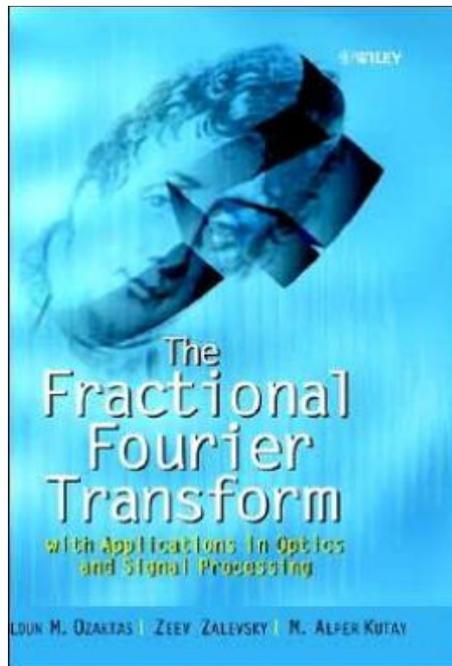
## Laplace transform of the above differential equation

$$G(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]} = \frac{b_1 s^{\gamma_1} + b_2 s^{\gamma_2} + \dots + b_m s^{\gamma_m}}{a_1 s^{\eta_1} + a_2 s^{\eta_2} + \dots + a_{n-1} s^{\eta_{n-1}} + a_n s^{\eta_n}}. \quad (7)$$

## Fourier transform of a fractional order derivative/integral operator

$$\mathcal{F}\left[-\infty D_t^\alpha f(t)\right] = (j\omega)^\alpha \mathcal{F}[f(t)], \quad (8)$$

where  $\alpha$  can be either positive or negative real number.



### The advantages of fractional order models

They depict the real world objects more accurately;  
Variable stability region;  
etc...

### G.W. Scott Blair (1950)

“We may express our concepts in Newtonian terms if **we find this convenient but**, if we do so, we must realize that we have made a translation into a language which is foreign to the system which we are studying.”



Integer-Order Calculus



Fractional-Order Calculus

Dr. Richard L. Magin, ICC12



Zhuo LI 2013

Why many integer order models depict real world objects well enough?

The “fractance” of many objects are usually very low.

Why integer order models are more popular?

Complexity of solutions to FO differential equations;

Ignorance of the existence of fractional order models.

Collection of resources for fractional order control

Fractional Order Controls: A beginner's view

<http://mechatronics.ucmerced.edu/sites/mechatronics.ucmerced.edu/files/page/documents/afc-csois-zhuo-2012-5-14.pdf>

Chen, Y.; Petras, I.; Xue, D, "[Fractional order control - A tutorial](#)" American Control Conference. ACC '09, pp.1397,1411, Jun 10-12, 2009

Podlubny, I, "[Fractional-Order Systems and  \$PI^\lambda D^\mu\$  Controllers](#)", IEEE Transactions on Automatic Control, Vol. 44, No. 1, January 1999

Oustaloup, Alan, etc, The CRONE Team, "[From Fractal Robustness to the CRONE Approach](#)", ESAIM: Proc., 1998, Vol. 5, pp. 177-192

## Stability of LTI Fractional Order Systems

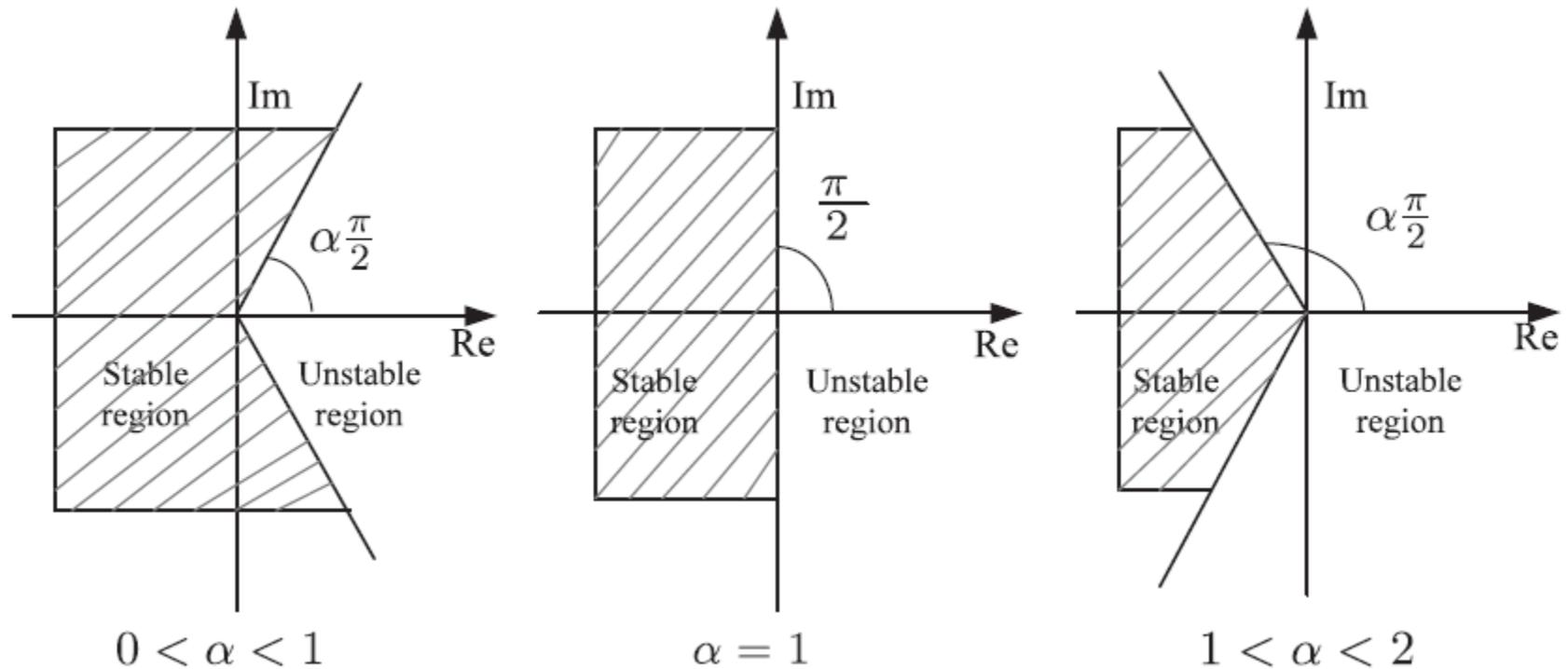


Figure: Stability region of LTI fractional order systems

### Four combinations

- IO model, IO controller
- **IO model, FO controller**
- **FO model, IO controller**
- **FO model, FO controller**

## TID Controller

Similar to PID control, the proportional compensating unit in TID (“Tilt” integral derivative) control is replaced with a compensator having a transfer function characterized by  $1/s^{1/n}$  or  $s^{-1/n}$ .

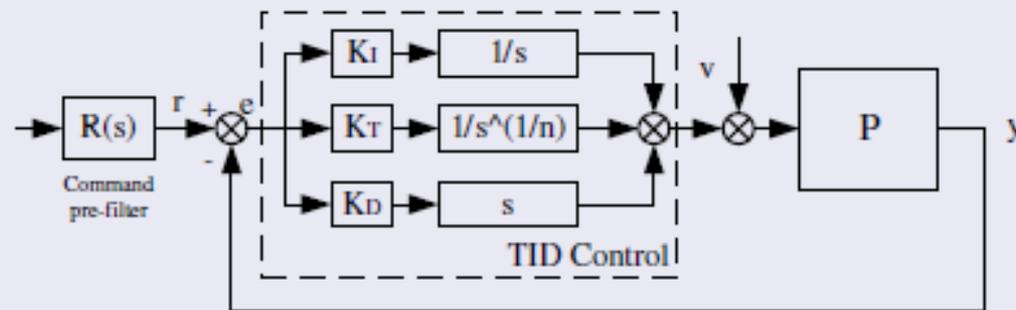
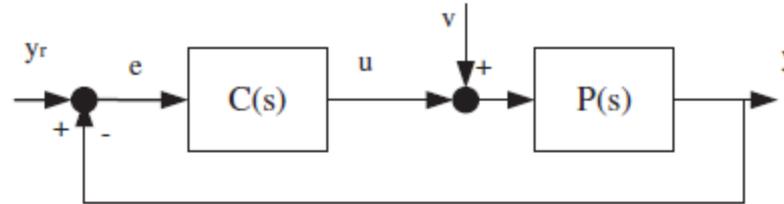
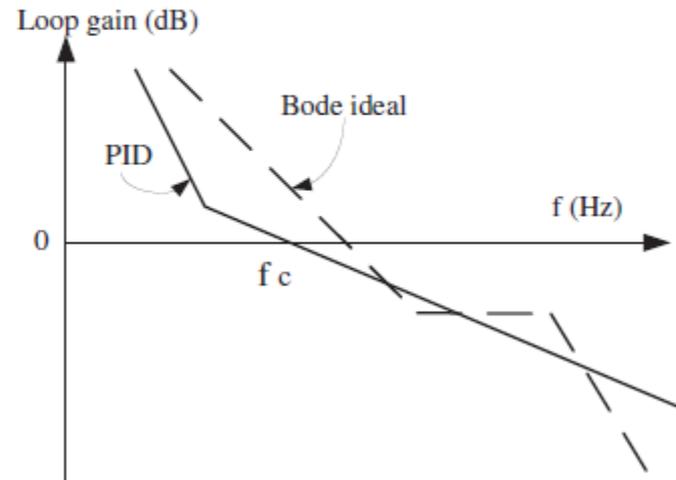


Figure: Block diagram of TID control scheme

## Motivation of the TID controller



(a) Block diagram of the classic feedback control system with disturbance



(b) Bode plots for PID controlled plant and the ideal loop response

Figure: Bode plots for PID controlled plant and the ideal loop response

### CRONE Controller

CRONE is a French abbreviation for “*Contrôle Robuste d’Ordre Non Entier*” (which means non-integer order robust control).

### Fractal Robustness

Used to describe the following two characteristics:

- Iso-damping lines;
- Frequency template.

PI<sup>λ</sup>D<sup>μ</sup> Controller

$$C(s) = \frac{U(s)}{E(s)} = K_p + T_i s^{-\lambda} + T_d s^{\delta}, \quad (10)$$

Better control performance can be expected due to the introduction of more tuning knobs.

## Tuning rule

- No systematic and rigorous method exist as for the conventional PID.
- A simple tuning scheme
  - $K_p > 100/E_t$ , where  $E_t$  is the static error;
  - Solve  $T_d$ ,  $\delta$ ,  $T_i$ ,  $\lambda$  from the characteristic equation with the FO controller

### Fractional Lead-Lag Compensator

$$C_r(s) = C_0 \left( \frac{1 + s/\omega_b}{1 + s/\omega_h} \right)^r, \quad (11)$$

where  $0 < \omega_b < \omega_h$ ,  $C_0 > 0$  and  $r \in (0, 1)$ .

### Design concept

- ideal loop transfer function:  $(\omega_0/j\omega)^n$  at  $|L_0(j\omega_0)| = 1$ ;
- try to shape the loop transfer function,  $L_0(j\omega)$ , close the the above shape, i.e.  $C(s) = C_r(s)G(s)$ , where  $G(s) \approx Ms^{-m}/P_0(s)$ .

### Realization

Stable minimum-phase frequency-domain fitting!

**This is the end of session I**

**Questions?**