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# A Tutorial on Fractional Order Motion Control

Part IV: Stability and Feasibility

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**MESA Lab** <http://mechatronics.ucmerced.edu>

- Fractional Order Motion Controls

John Wiley & Sons, Inc.

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- Dr. Ying Luo

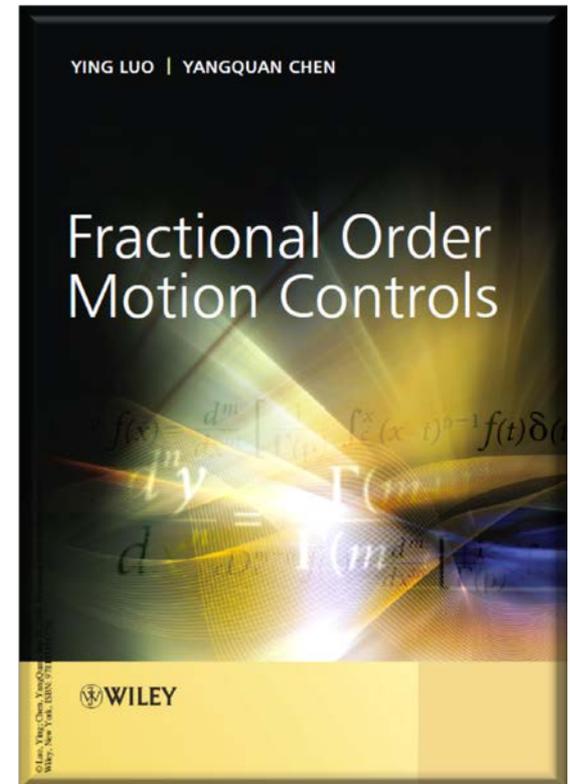
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- PART I FUNDAMENTALS OF FRACTIONAL CONTROLS
- 1 Introduction 3
  
- PART II FRACTIONAL ORDER VELOCITY SERVO
- 2 Fractional Order PI Controller Designs for Velocity Servo Systems 25
- 3 Tuning Fractional Order PI Controllers for Fractional Order Velocity Systems with Experimental Validation 41
- 4 Relay Feedback Tuning of Robust PID Controllers 59
- 5 Auto-Tuning of Fractional Order Controllers with ISO-Damping 73
  
- PART III FRACTIONAL ORDER POSITION SERVO
- 6 Fractional Order PD Controller Tuning for Position Systems 91
- 7 Fractional Order [PD] Controller Synthesis for Position Servo Systems 105
- 8 Time-Constant Robust Analysis and Design of Fractional Order [PD] Controller 123
- 9 Experimental Study of Fractional Order PD Controller Synthesis for Fractional Order Position Servo Systems 139
- 10 Fractional Order [PD] Controller Design and Comparison for Fractional Order Position Servo Systems 155
  
- **PART IV STABILITY AND FEASIBILITY FOR FOPID DESIGN**
- **11 Stability and Design Feasibility of Robust PID Controllers for FOPTD Systems 165**
- **12 Stability and Design Feasibility of Robust FOPI Controllers for FOPTD Systems 187**
  
- PART V FRACTIONAL ORDER DISTURBANCE COMPENSATORS
- 13 Fractional Order Disturbance Observer 211
- 14 Fractional Order Adaptive Feed-forward Cancellation 223
- 15 Fractional Order Robust Control for Cogging Effect 243
- 16 Fractional Order Periodic Adaptive Learning Compensation 275
  
- PART VI EFFECTS OF FRACTIONAL ORDER CONTROLS ON NONLINEARITIES
- 17 Fractional Order PID Control of A DC-Motor with Elastic Shaft 293
- 18 Fractional Order Ultra Low-Speed Position Servo 313
- 19 Optimized Fractional Order Conditional Integrator 329
  
- PART VII FRACTIONAL ORDER CONTROL APPLICATIONS
- 20 Lateral Directional Fractional Order Control of A Small Fixed-Wing UAV 345
- 21 Fractional Order PD Controller Synthesis and Implementation for HDD Servo System 369

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- 8 Time-Constant Robust Analysis and Design of Fractional Order [PD] Controller 123
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## Plant

$$P(s) = \frac{K}{Ts + 1} e^{-Ls}$$

## Controller

$$C(s) = K_p + \frac{K_i}{s} + K_d s$$

## The gain-phase margin tester

$$M_T(A, \phi) = A e^{-j\phi}$$

A is the boundary of gain margin, is the boundary of phase margin

**Definition: Gain-phase margin tester**

$M_T$  in figure below is a gain-phase master which provides information for plotting the boundaries of constant gain margin and phase margin in the parameter plane.

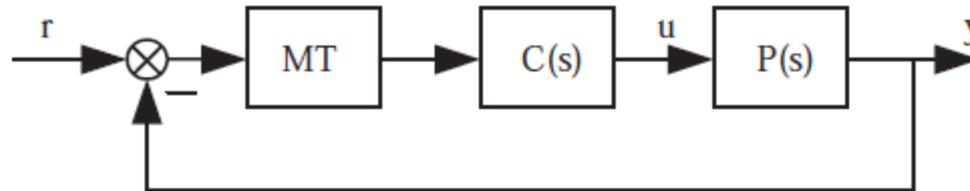


Figure: The feedback control system with the gain-phase margin tester

## Open loop

$$G(s) = M_T(A, \phi)C(s)P(s)$$

## Closed loop

$$\begin{aligned}\Phi(s) &= \frac{M_T(A, \phi)C(s)P(s)}{1 + M_T(A, \phi)C(s)P(s)} \\ &= \frac{Ae^{-j\phi}Ke^{-Ls}(K_p s + K_i + K_d s)}{s(Ts + 1) + Ae^{-j\phi}Ke^{-Ls}(K_p s + K_i + K_d s)}\end{aligned}$$

## Property

Assuming  $\phi = 0$  for  $M_T$ , the controller parameters can be obtained satisfying a given gain margin  $A$ . Vice versa, assuming  $A = 1$  for  $M_T$ , one can obtain the controller parameters for a given phase margin  $\phi$ .

Definition: **IRB** (Infinity root boundary):

$$D(K_p, K_i, K_d, A, \phi; s = \infty) = 0$$

$$\Rightarrow K_d = \pm \frac{T}{AK}$$

Definition: **RRB** (Real root boundary):

$$D(K_p, K_i, K_d, A, \phi; s = 0) = 0$$

$$\Rightarrow K_i = 0$$

Definition: **CRB** (Complex root boundary):

$$D(K_p, K_i, K_d, A, \phi; s = j\omega) = 0$$

$$\Rightarrow \begin{cases} K_d = \frac{KA \sin(\phi + \omega L) K_i - KA \omega K_p \cos(\phi + \omega L) - \omega}{KA \omega^2 \sin(\phi + \omega L)} \\ K_p = \frac{T \omega \sin(\phi + \omega L) - \cos(\phi + \omega L)}{KA} \\ K_i = \frac{\omega \sin(\phi + \omega L) - T \omega^2 \cos(\phi + \omega L)}{KA} + \omega^2 K_d \end{cases}$$

## Property

The controller parameter boundaries of the stability region Q can be determined by the infinity root boundary (IRB), the real root boundary (RRB), and the complex root boundary (CRB)

$$k_d = \begin{cases} 0 & \text{for } (\alpha_n = \beta_n) \\ & \text{or } (\alpha_n > \beta_n \text{ and } \mu > \alpha_n - \beta_n) \\ \pm a_n/b_n & \text{for } (\alpha_n > \beta_n \text{ and } \mu = \alpha_n - \beta_n) \\ \text{none} & \text{for } (\alpha_n > \beta_n \text{ and } \mu < \alpha_n - \beta_n). \end{cases} \quad (8)$$

**An Algorithm for Stabilization of Fractional-Order Time Delay Systems Using Fractional-Order PID Controllers**

Serdar Ethem Hamamci

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. 52, NO. 10,  
OCTOBER 2007

**Definition: the relative stability line and surface**

Given one specification – phase margin  $\varphi = \varphi_m$  ( $A = 1$ ), and from CRB, a relative stability line can be drawn in the  $(K_i, K_d)$  parameter-plane as  $\omega \rightarrow \omega_0$  from zero with a certain fixed  $K_{d1} \in [-T/K, T/K]$ .  $\omega_0$  is the maximum frequency guaranteeing the pre-specified phase margin with the fixed  $K_{d1}$  on the relative stability line. Sweeping all the  $K_d$  in  $[-T/K, T/K]$ , a surface in the 3D parameter-space can be generated satisfying the pre-specified phase margin  $\varphi_m$ , which is called the relative stability surface. Since there exists a maximum frequency  $\omega_0$  for every relative stability line, the frequency boundary of the relative stability surface can be found.

**Property**

The frequency  $\omega$  of every point on the relative stability line can be used as the gain crossover frequency with the corresponding PID controller parameters

### Definition: the flat phase stable point

All the points with PID parameters on the relative stability curve can be tested by the equation  $\frac{d\phi}{d\omega} = 0$ . If a certain point with the PID parameters  $(K_p, K_i, K_d)$  can be found to guarantee the relationship, this point is called the flat phase stable point.

### Property

With this flat phase constraint, the open-loop system phase can maintain almost the same value when the loop gain changes in a certain interval, namely, the system with this designed PID is robust to the loop gain variations. The overshoots of the step responses are almost the same with the variations of loop gain in certain range. Hence, the control performance of the system with the designed PID controller degrades gracefully when the steady-state gains of the plant and the PID controller change.

## Example

$$K_d = 0.5$$

Complete stability region

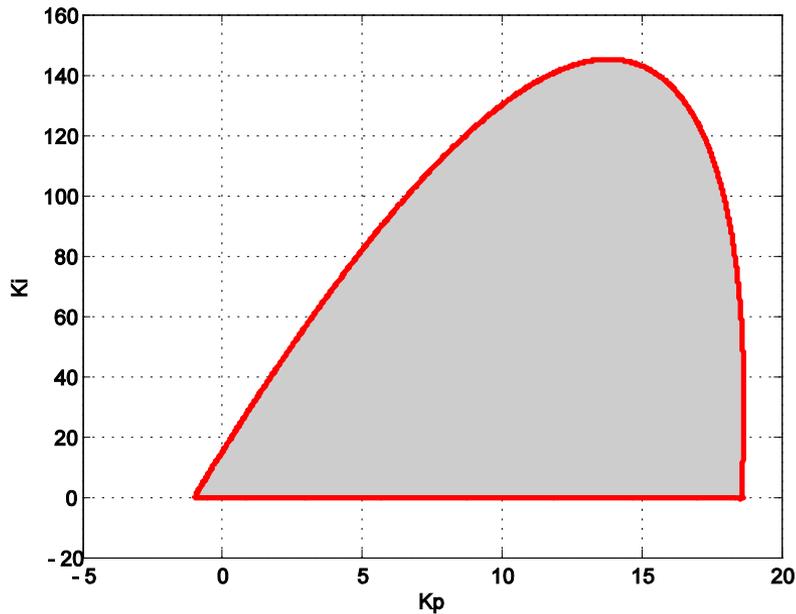
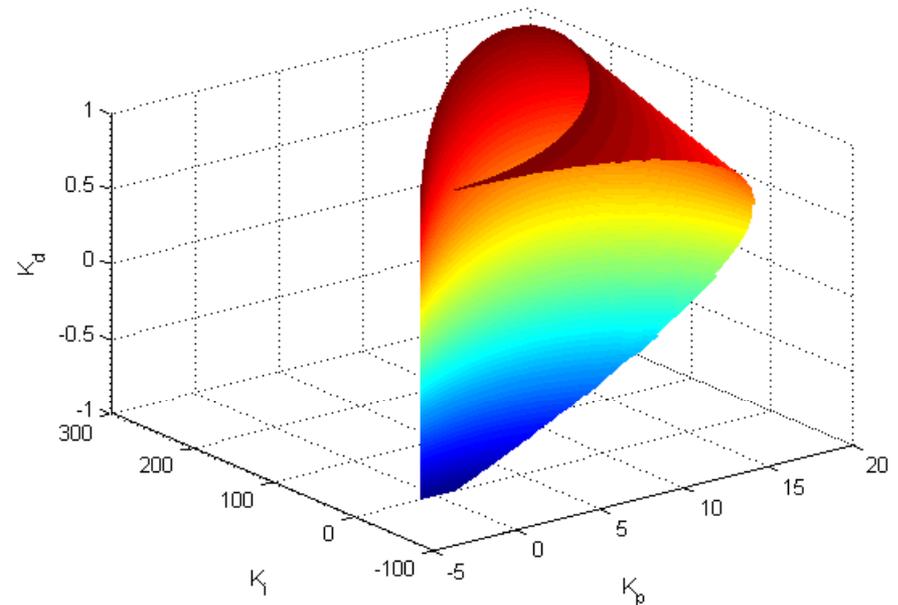


Figure: Stability region of  $K_i$  with respect to  $K_p$  with  $K_d = 0.5$



Complete stability region of  $K_i$ ,  $K_p$  and  $K_d$

## Example

$$K_d = 0.5$$

$$\varphi_m = 50^\circ$$

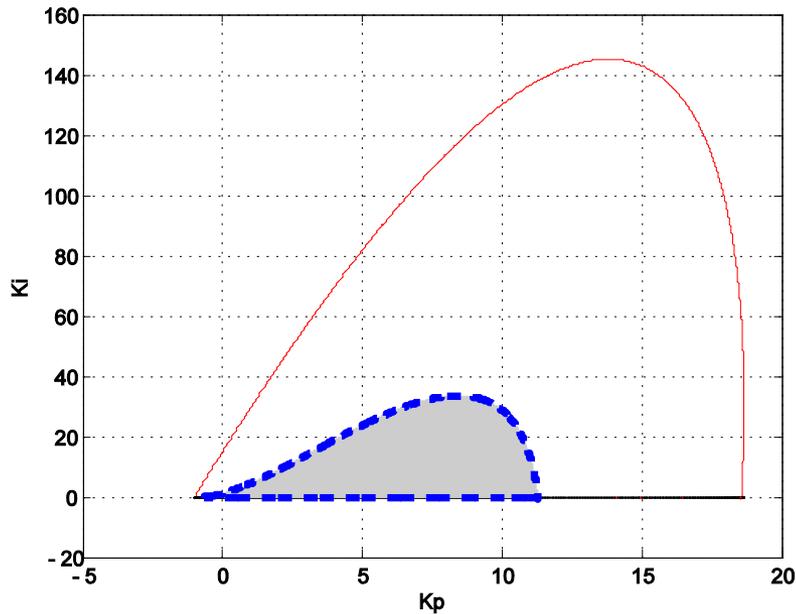


Figure: Stability region of  $K_i$  with respect to  $K_p$  with  $K_d = 0.5$  and  $\varphi_m = 50^\circ$

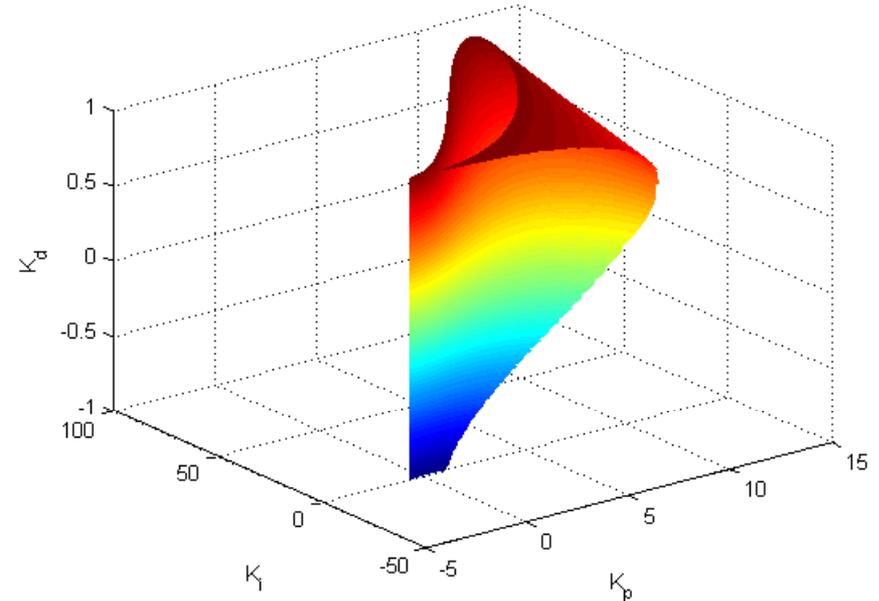
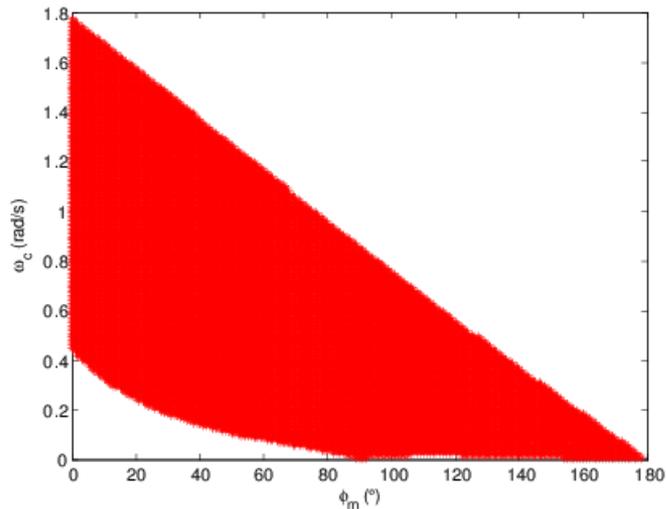
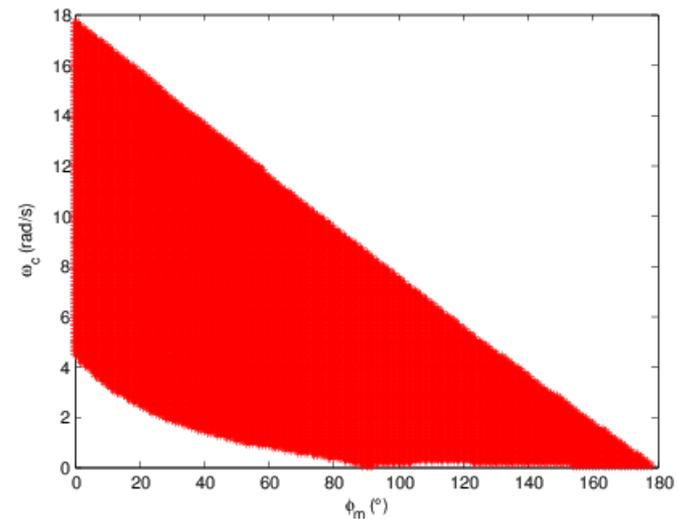
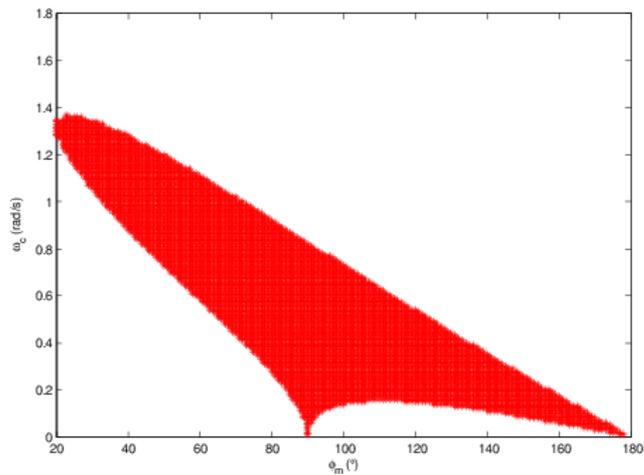
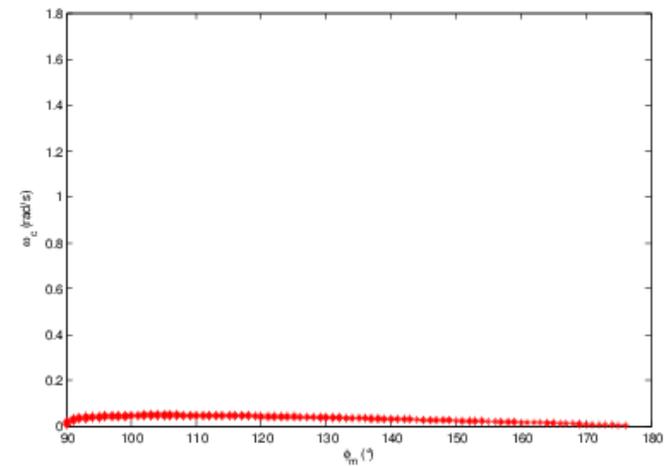


Figure: Complete stability region of  $K_i$ ,  $K_p$  and  $K_d$  and  $\varphi_m = 50^\circ$

## Example: Achievable region for PID

 $T = 1\text{s}$  and  $L = 0.1\text{s}$  $T = 10\text{s}$  and  $L = 1\text{s}$  $T = 1\text{s}$  and  $L = 1\text{s}$  $T = 1\text{s}$  and  $L = 10\text{s}$

## The plant

$$P(s) = \frac{1}{s + 1} e^{-0.1s}$$

## The PID controllers designed by different tuning methods

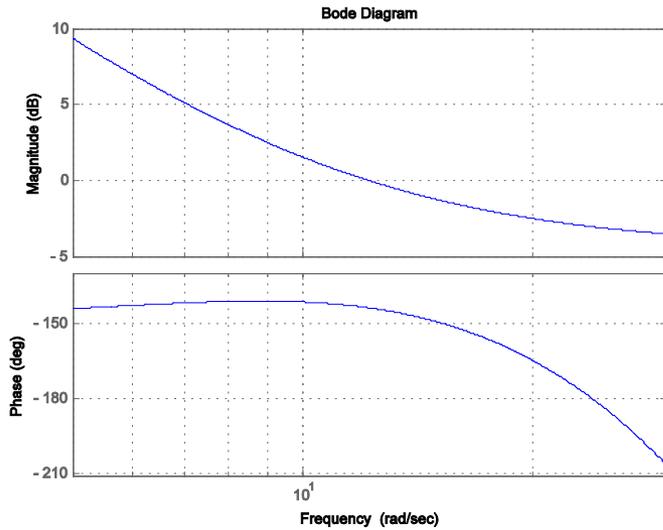
Ziegler-Nichols

$$K_p = \frac{1.2T}{KL}, K_i = \frac{K_p}{2L}, K_d = \frac{K_p L}{L}$$

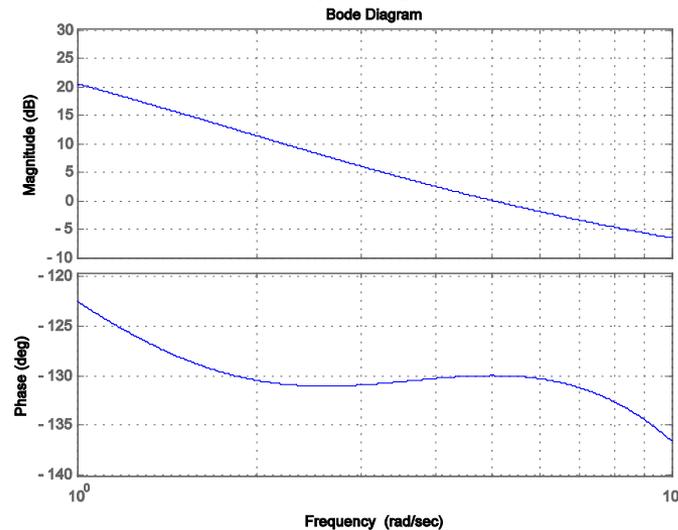
$$C(s) = 12 + \frac{60}{s} + 0.6s$$

The flat phase constraint

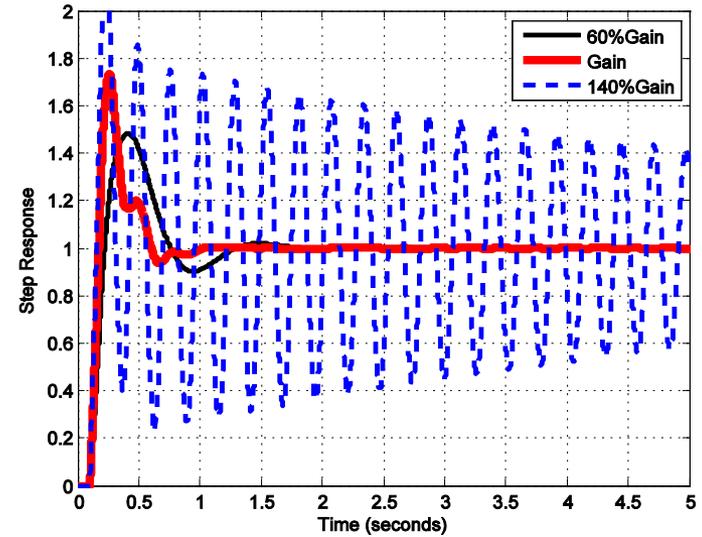
$$C(s) = 4.71 + \frac{14.48}{s} + 0.19s$$



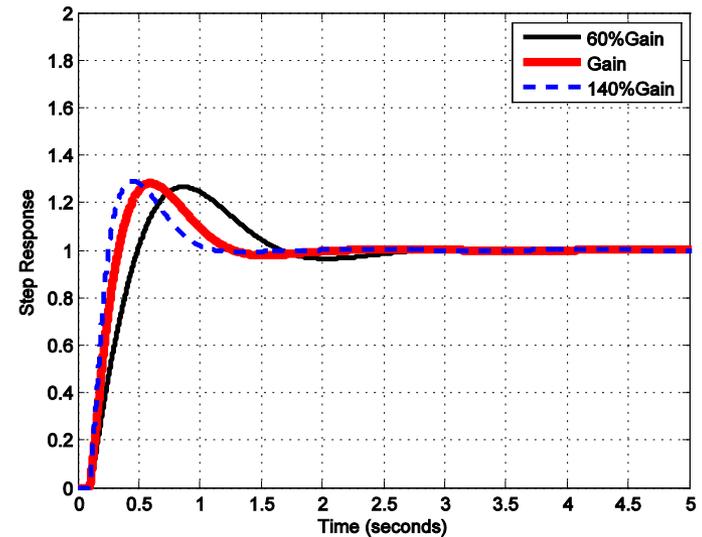
*Bode plot of ZN PID*



*Bode plot of flat phase PID*



*Step response of ZN PID*



*Step response of flat phase PID*

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The controller

$$C(s) = K_p + \frac{K_i}{s^r}$$

the characteristic equation of the closed-loop system

$$D(K_p, K_i, A, \phi; s) = s^r (Ts + 1) + Ae^{-j\phi} Ke^{-Ls} (K_p s^r + K_i)$$

## RRB

$$D(K_p, K_i, K_d, A, \phi; s = 0) = 0$$

$$\Rightarrow K_i = 0$$

## CRB

$$D(K_p, K_i, K_d, A, \phi; s = j\omega) = 0$$

$$\begin{aligned} D(K_p, K_i, r, A, \phi; j\omega) &= (j\omega)^r (jT\omega + 1) + Ae^{-j\phi} e^{-j\omega L} K (K_p (j\omega)^r + K_i) \\ &= \omega^r \cos \frac{r\pi}{2} - T\omega^{1+r} \sin \frac{r\pi}{2} \end{aligned}$$

$$\begin{aligned} \Rightarrow & + AK \cos(\phi + \omega L) (K_i + K_p \omega^r \cos r\pi/2) + AK \sin(\phi + \omega L) K_p \omega^r \sin \frac{r\pi}{2} \\ & + j(T\omega^{1+r} \cos \frac{r\pi}{2} + \omega^r \sin \frac{r\pi}{2} + AK \cos(\phi + \omega L) K_p \omega^r \sin \frac{r\pi}{2} \\ & - AK \sin(\phi + \omega L) (K_i + K_p \omega^r \cos r\pi/2)) \\ & = 0, \end{aligned} \tag{12}$$

CRB – cont.

⇒

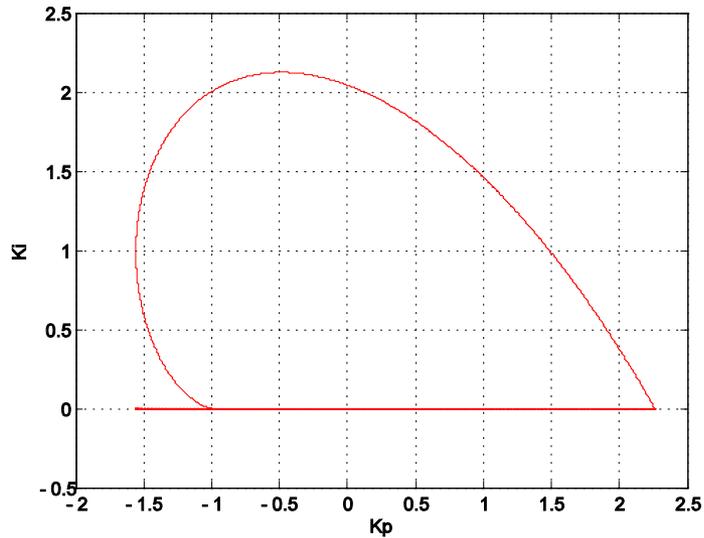
$$K_p = \frac{-(B_1 S_1 + B_2 C_1)}{AK S_2 \omega^r},$$

$$K_i = \frac{B - B_1 S_1 C_1 - B_2 C_1^2}{AK S_1} + \frac{B_1 S_1 C_2 + B_2 C_1 C_2}{AK S_2}.$$

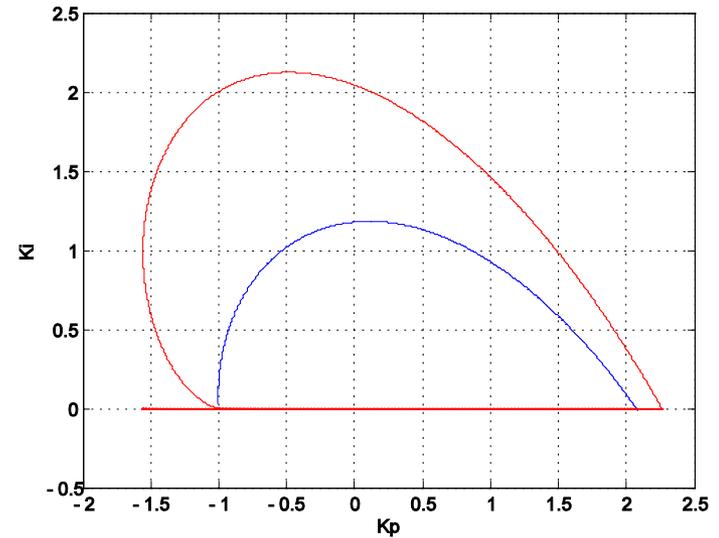
With plat phase constraint

$$\frac{d\phi}{d\omega} = \frac{(B_1^2 + B_2^2)(EF' - E'F) + (B_1' B_2 - B_1 B_2')(E^2 + F^2)}{(B_1 E + B_2 F)^2 + (B_1 F - B_2 E)^2} - L$$

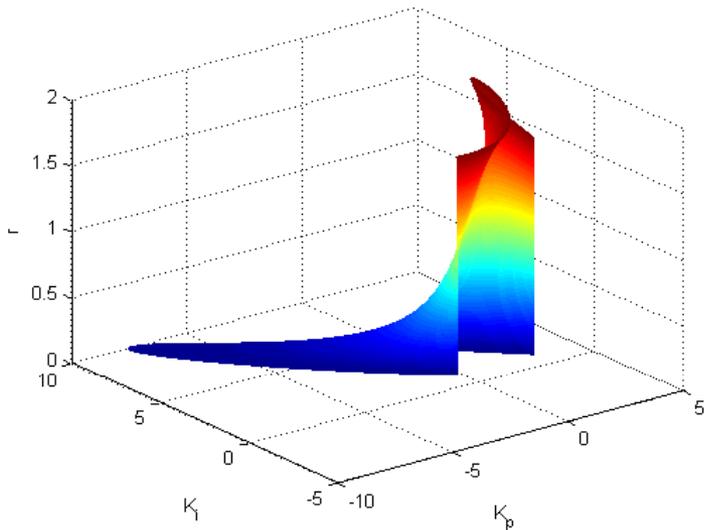
$$= 0,$$



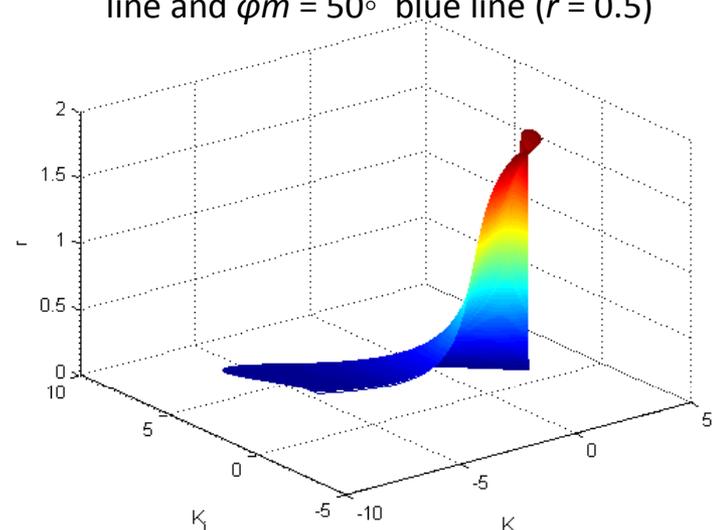
Stability boundary of  $K_i$  vs.  $K_p$  with  $r = 0.5$



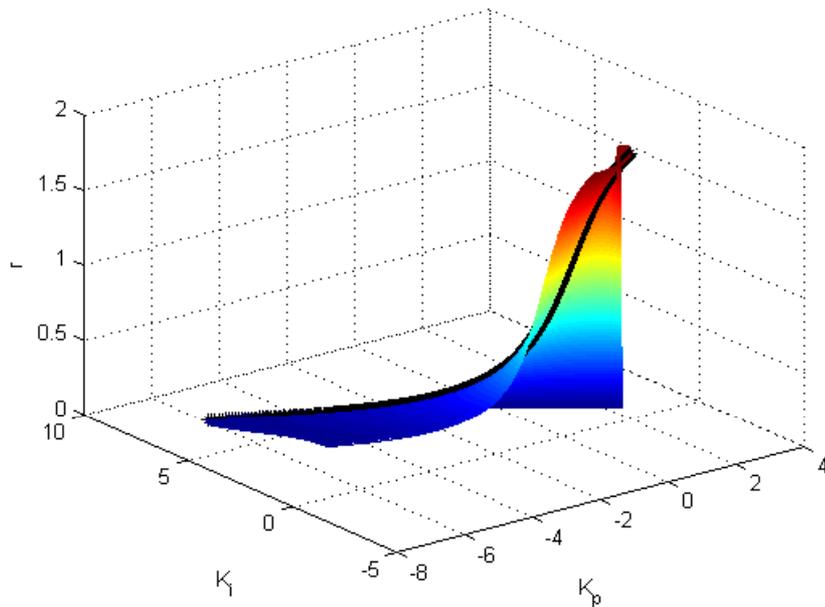
Stability region comparison of  $K_i$  vs.  $K_p$  with  $\phi_m = 0^\circ$  dark line and  $\phi_m = 50^\circ$  blue line ( $r = 0.5$ )



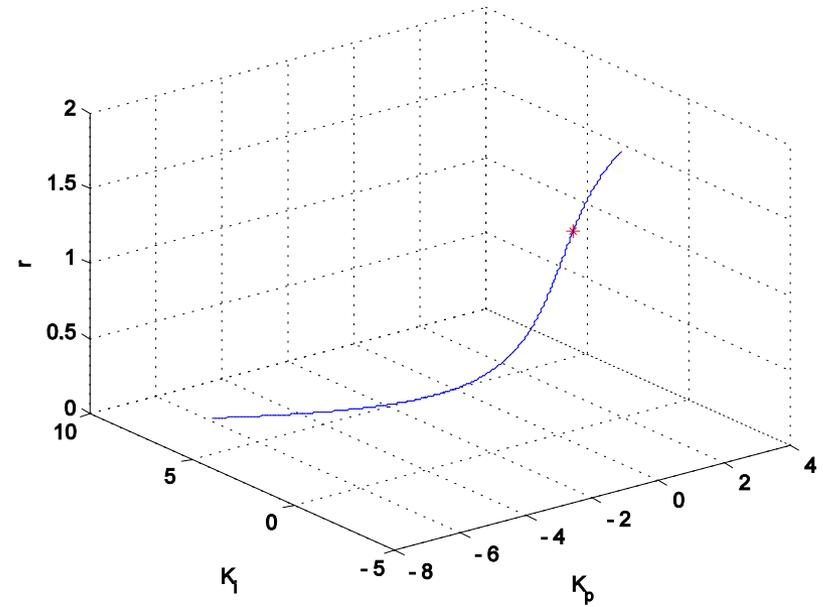
Complete stability region of  $K_i$ ,  $K_p$  and  $r$



Three dimensional relative stability surface of  $K_i$ ,  $K_p$  and  $r$  with  $\phi_m = 50^\circ$



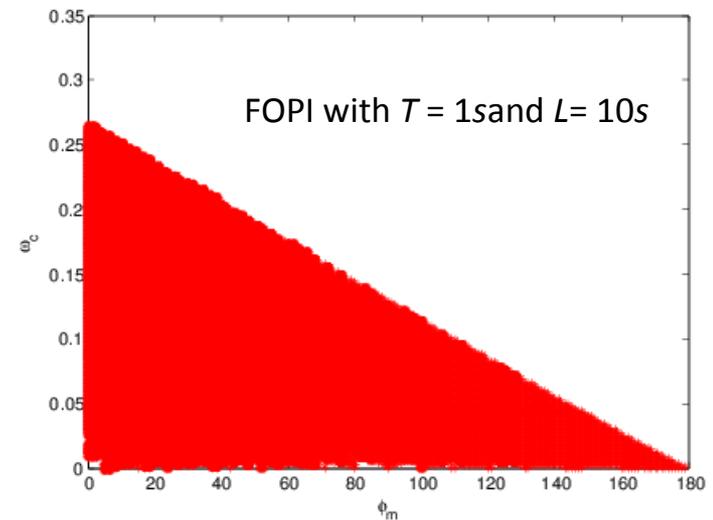
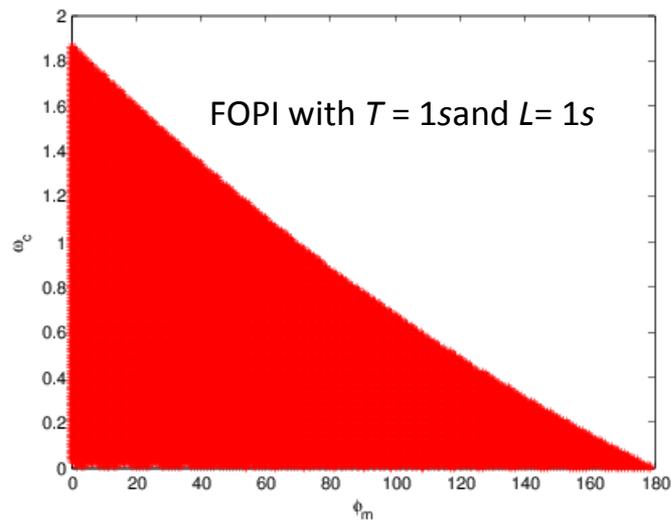
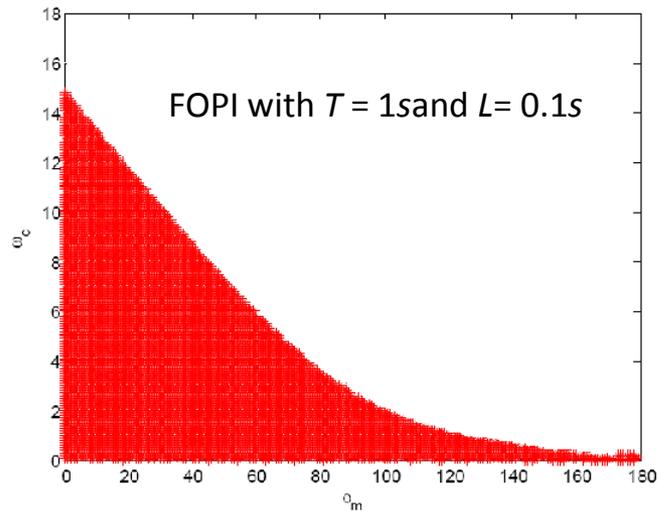
The relative stability curve on the relative stability surface



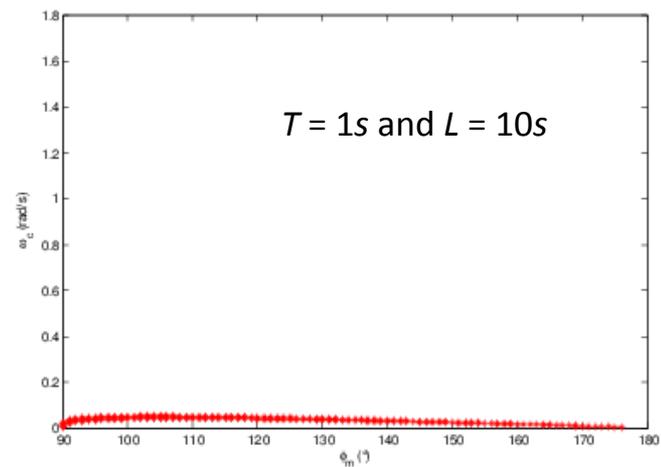
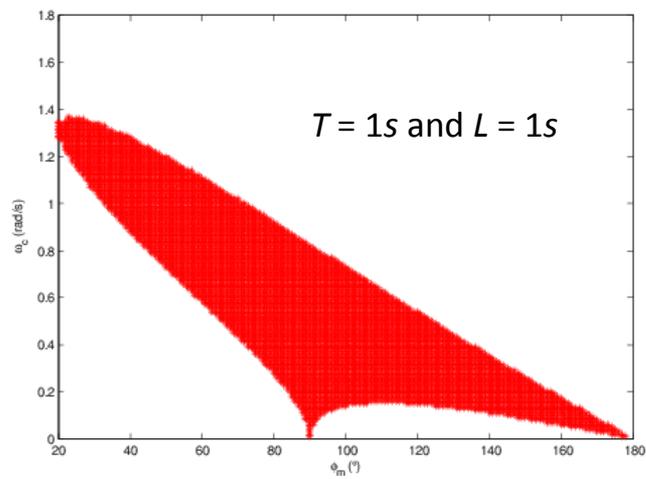
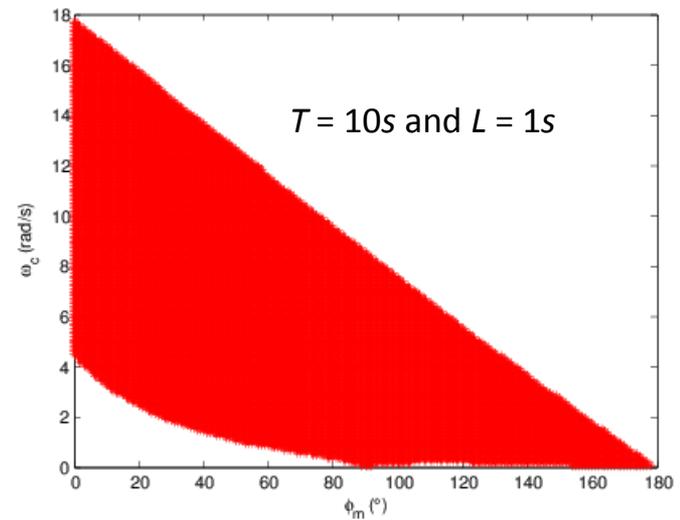
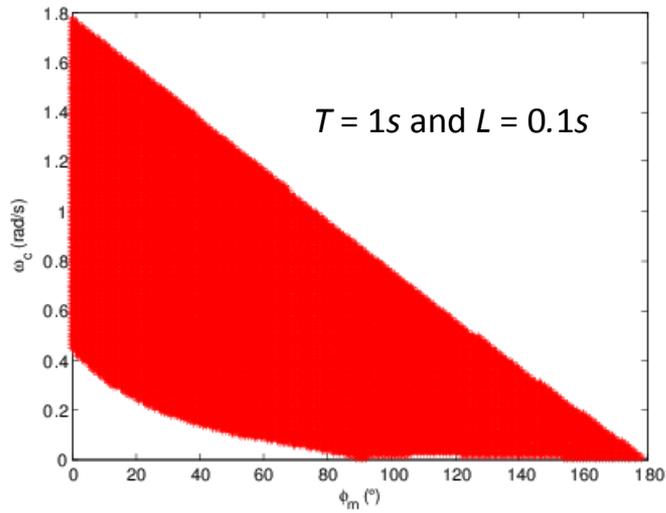
The flat phase stable point on the relative stability curve

Figure: The relative stability curve and the flat phase stable point in the 3D parameter space

## Example: Achievable region for FOPI



## Recall: Achievable region for IOPID



**Conclusion**

FO PI has larger achievable region than IOPID

## The plant

$$P(s) = \frac{1}{s + 1} e^{-0.1s}$$

## The PID controllers designed by different tuning methods

AMIGO (Approximate M-constrained integral gain optimization)

$$K_p = \frac{1.5}{K} + \left[ 0.35 - \frac{LT}{(L+T)^2} \right] \frac{T}{KL}$$

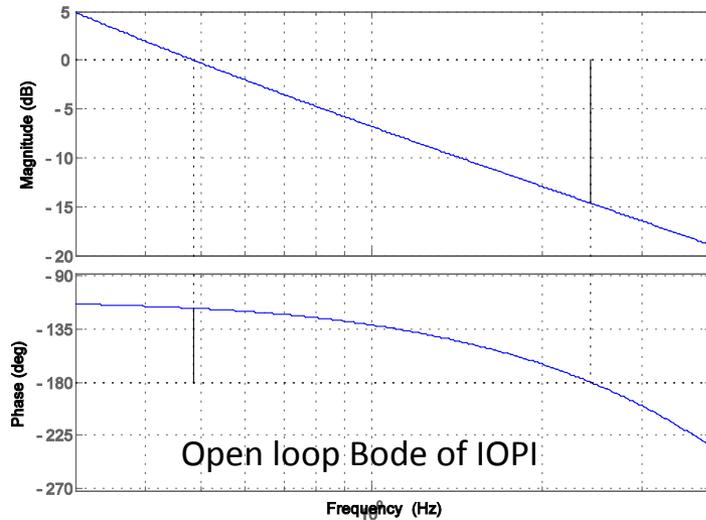
$$K_i = \frac{K_p}{2L} \left( 0.35L + \frac{13LT^2}{T^2 + 12LT + L^2} \right)$$

$$C(s) = 2.82 + \frac{4.65}{s}$$

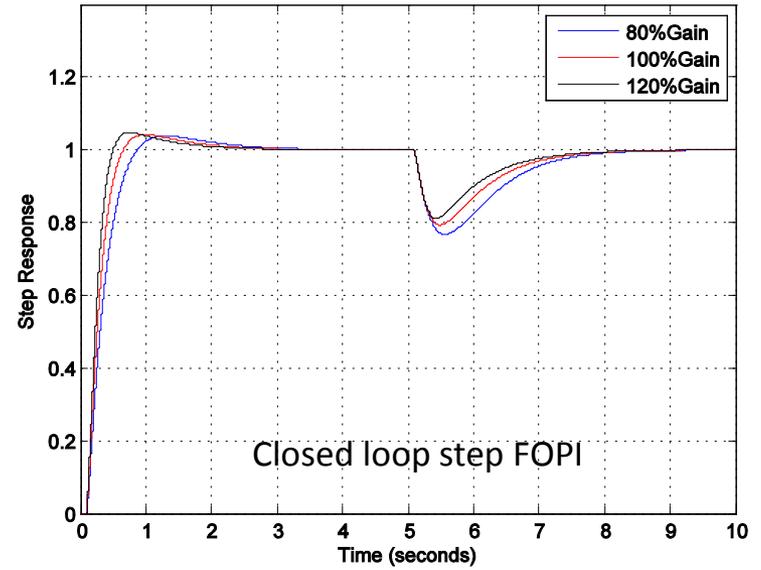
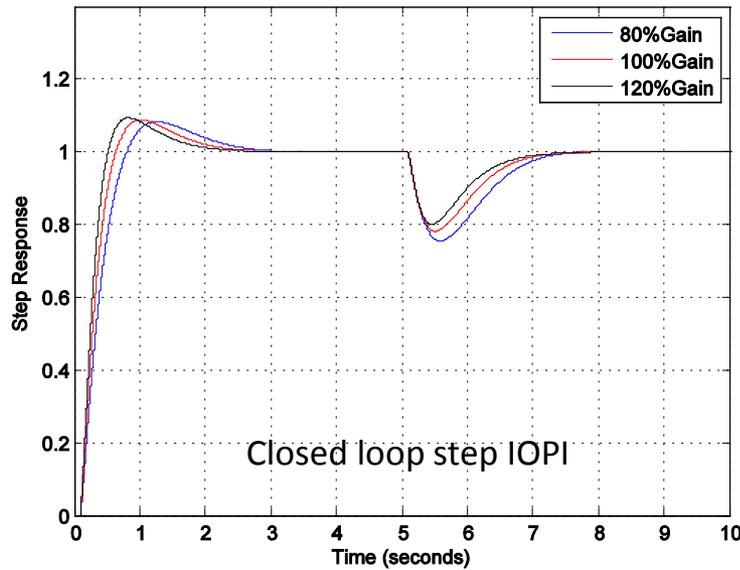
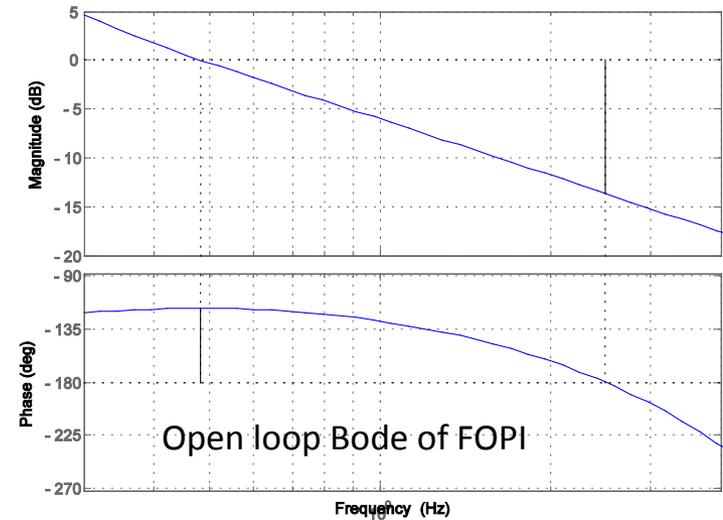
The flat phase constraint

$$C(s) = 3.34 + \frac{6.20}{s^{1.44}}$$

Bode Diagram  
Gm = 14.6 dB (at 2.43 Hz) , Pm = 62.3 deg (at 0.485 Hz)



Bode Diagram  
Gm = 13.6 dB (at 2.5 Hz) , Pm = 62.3 deg (at 0.484 Hz)



**This is the end of session IV**

**Questions?**